1. (a) One can show that if \( G : \mathbb{R}^1 \to \mathbb{R}^1 \) is Lipschitz continuously differentiable near \( x_* \in \mathbb{R}^1 \) such that \( x_* = G(x_*) \) and \( G'(x_*) = 0 \), then for \( x_0 \) near \( x_* \), the fixed-point iterates \( \{x_n\} \) converge quadratically to \( x_* \). Use this to show that if \( p \) is any positive number, then \( x_* = \sqrt{p} \) satisfies the fixed-point equation

\[
x = G(x) = \frac{1}{2} x + \frac{p}{2x}
\]

and, moreover, the local convergence of fixed-point iterates to \( \sqrt{p} \) is quadratic.

(b) A symmetric positive-definite (SPD) matrix \( A \in \mathbb{R}^{n \times n} \) has a unique SPD square root, i.e., a unique SPD \( M \in \mathbb{R}^{n \times n} \) such that \( A = M^2 \). One easily verifies that \( M \) satisfies the analog of (1):

\[
M = G(M) = \frac{1}{2} \{M + M^{-1}A\} .
\]

Fixed-point iteration based on \( G \) can sometimes (see below) be used to find the SPD square root of an SPD \( A \). Apply this fixed-point iteration starting with \( M = I \in \mathbb{R}^{5 \times 5} \) to find an approximate SPD square root of

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 2
\end{pmatrix} .
\]

At each iteration, print out the iteration number and the norm of \( A - M^2 \). (In MATLAB, you can use \texttt{"norm(A-M*M)"}, which produces \( \|A - M^2\|_2 \).) Terminate the iterations when the norm of \( A - M^2 \) is less than \( 10^{-10} \). Print out the final \( M \).

Remarks:

(i) The existence of the SPD square root \( M \) of an SPD matrix \( A \) can be shown in several ways. Perhaps the easiest way is to consider the canonical form \( A = U\Lambda U^T \), where \( U \) is orthogonal \( (U^TU = I) \) and \( \Lambda \) is a diagonal matrix with the (positive) eigenvalues of \( A \) on its diagonal. Then \( M = U\Lambda^{1/2}U^T \), where \( \Lambda^{1/2} \) is the diagonal matrix with diagonal entries equal to the square roots of those of \( \Lambda \).

(ii) Fixed-point iteration using \( G \) in (2) is not a good method for finding the SPD square root. It is often unstable and is expensive relative to alternatives. Perhaps the best way is based on the observations in (i): Compute the decomposition \( A = U\Lambda U^T \), which amounts to computing an orthonormal set of eigenvectors and their corresponding eigenvalues, and then take \( M = U\Lambda^{1/2}U^T \). MATLAB’s \texttt{sqrtm} algorithm uses this approach.
(iii) In implementing the fixed-point iteration in MATLAB, the term $M^{-1}A$ should be evaluated using $\texttt{M}\backslash A$, not $\texttt{inv(M)}*A$.

(iv) The matrix $A$ in (3) arises in discretizing the second-derivative operator $d^2/dx^2$ using finite differences. It is SPD, although that is not immediately apparent.

2. (20 points) An object falling vertically through the air is subjected to not only the force of gravity but also viscous resistance from the air. Let $s(t)$ denote the height of the object above the ground. Newton’s law $F = ma$ leads to the differential equation $ms'' = -g - ks'$, where $m$ is the mass of the object, $g$ is the acceleration of gravity, and $k > 0$ is the coefficient of air resistance. If the object is dropped from rest (zero velocity) from an initial height $s_0$, then solving the resulting initial value problem yields

$$s(t) = s_0 - \frac{mg}{k} t + \frac{m^2 g}{k^2} \left(1 - e^{-kt/m}\right).$$

Setting $s(t) = 0$ results in a nonlinear equation for the time of impact, and we can rearrange it to obtain the fixed-point equation

$$t = G(t) \equiv \frac{ks_0}{mg} + \frac{m}{k} \left(1 - e^{-kt/m}\right).$$

(a) Show that there is a unique $t > 0$ that satisfies this equation and that fixed-point iteration converges to it for any initial $t_0 > 0$. Hint: Show that, for any $t > 0$, $\frac{ks_0}{mg} \leq G(t) \leq \frac{ks_0}{mg} + \frac{m}{k}$ and $0 < G'(t) < 1$. Then consider any interval $[a, b]$ with $0 < a \leq \frac{ks_0}{mg} < \frac{ks_0}{mg} + \frac{m}{k} \leq b$, and apply a theorem given in class.

(b) Take $s_0 = 300$ ft, $m = 0.25$ lb, $k = 0.1$ lb.-sec/ft, and $g = 32.17$ ft/sec$^2$. Apply fixed-point iteration with $G$ to find an approximate fixed point $t$ such that $|t - G(t)| \leq 10^{-10}$. At each iteration, print out the iteration number, the current approximate solution $t$, and $|G(t) - t|$.

Remarks: This adapted from problem 23, p. 63, of R. L. Burden and J. D. Faires, “Numerical Analysis”, 8th edition, Thomson–Brooks/Cole, 2005. Note that expressing the force resulting from air resistance as $-ks'$ amounts to assuming that air resistance depends linearly on the velocity. This is valid over small ranges of the velocity but not over large ones.