1. Suppose $A \in \mathbb{R}^{m \times n}$, $m \geq n$, and $A = QR$, where $Q \in \mathbb{R}^{m \times n}$ is orthogonal, i.e., $Q^TQ = I$, and $R \in \mathbb{R}^{n \times n}$ is upper triangular and nonsingular. Show the following:

(a) $\|Qv\|_2 = \|v\|_2$ for all $v \in \mathbb{R}^n$.

(b) $\kappa_2(A) = \kappa_2(R)$.

Recall: If $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, then

$$\|A\|_2 = \max_{\|v\|_2 = 1} \|Av\|_2$$

and

$$\|A\|_2 = \min_{\|v\|_2 = 1} \|Av\|_2.$$

2. Consider the following values of $(t_i, y_i)$, $i = 1, \ldots, 11$:

$$(0.0, 0.998), (0.3, 0.623), (0.6, 0.364), (0.9, 0.100), (1.2, 0.270), (1.5, 0.153),$$

$$(1.8, 0.362), (2.1, 0.375), (2.4, 0.146), (2.7, 0.033), (3.0, -0.509).$$

Suppose we want to approximate these data with a cubic polynomial $y = x_1 + x_2t + x_3t^2 + x_4t^3$ using the method of least-squares to determine the coefficients $x_1, \ldots, x_4$. Then $x = (x_1, \ldots, x_4)^T$ is chosen to minimize the sum of squared residuals

$$R(x) = \sum_{i=1}^{11} (y_i - x_1 - x_2t_i - x_3t_i^2 - x_4t_i^3)^2 = \|b - Ax\|_2^2,$$

for an appropriate matrix $A$ and vector $b$.

(a) Form the normal equations of least-squares $A^TAx = A^Tb$ and solve them using Cholesky factorization. Print out and hand in the $4 \times 4$ matrix $A^TA$, the computed solution $x$, and the estimated condition number of $A^TA$ obtained using MATLAB’s “cond” command. You may use the MATLAB “chol” command to obtain the Cholesky factors and the “\” (backslash) command for forward- and back-substitution solves with the factors. (MATLAB is smart enough to know that a triangular matrix is triangular and to use forward- or back-substitution for solves without factoring.)

(b) Solve the least-squares problem using the QR decomposition $A = QR$. Print out and hand in the $11 \times 4$ matrix $A$, the computed solution $x$, and the estimated condition number of $A$ obtained using MATLAB’s “cond” command. You may use the MATLAB “qr” command to obtain the QR decomposition.

(c) Use MATLAB’s “plot” command to plot the given data along with a graph over $[0, 3]$ of the cubic polynomial determined by $x$ in either (a) or (b). In plotting the graph of the cubic, use enough points so that it appears as a smooth curve (not a piecewise-linear curve). Typing “help plot” or “doc plot” will provide information on how to plot the data as discrete points. The “hold” command may be helpful in plotting the data and the graph of the cubic in a single figure.