Gaussian Elimination with Partial Pivoting

The following algorithms implement Gaussian elimination with partial pivoting followed by back substitution to compute the solution of $Ax = b$, where $A$ is an $n \times n$ matrix with $ij$th entry $a_{ij}$ and $b$ is an $n$-vector with $i$th component $b_i$.

Like our naive Gaussian elimination algorithms, these are structured as most modern software library routines are. The first algorithm performs Gaussian elimination with partial pivoting on $A$, overwriting the lower triangular part of $A$ with the “multipliers”. The second algorithm performs the same pivoting and row operations on $b$ that were performed on $A$. The third algorithm performs back substitution, overwriting $b$ with the solution. In library software, the first algorithm is usually coded in a routine separate from the other two, and the second and third algorithms are usually combined in one routine that uses the output of the first routine. These algorithms require no more storage than is required for $A$ and $b$, except an additional vector of length $n - 1$ may be required to pass pivot information (the $i_k$'s) from the first routine to the second.

**Gaussian Elimination with Partial Pivoting on $A$:**

For $k = 1, \ldots, n - 1$

Find $i_k \geq k$ such that $|a_{i_k k}| = \max_{k \leq i \leq n} |a_{ik}|$.

If $i_k > k$, interchange $a_{kj} \leftrightarrow a_{i_k j}$ for $j = k, \ldots, n$.

For $i = k + 1, \ldots, n$

$a_{ik} \leftarrow a_{ik} / a_{kk}$

For $j = k + 1, \ldots, n$

$a_{ij} \leftarrow a_{ij} + a_{ik} a_{kj}$

**Row Operations on $b$:**

For $k = 1, \ldots, n - 1$

If $i_k > k$, interchange $b_k \leftrightarrow b_{i_k}$.

For $i = k + 1, \ldots, n$

$b_i \leftarrow b_i + a_{ik} b_k$

**Back Substitution:**

$b_n \leftarrow b_n / a_{nn}$

For $i = n - 1, \ldots, 1$

$b_i \leftarrow (b_i - \sum_{j=i+1}^{n} a_{ij} b_j) / a_{ii}$

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1. The MATLAB “backslash” command, which produces the solution of $Ax = b$ with the single command $A \backslash b$, appears to be an exception. In fact, it separates the operations on $A$ from those on $b$ as above “under the covers.”