In working the following exam, use no books or notes. *Show your work* on all but the first problem. Use the backs of the pages if you need more space.

1. (10 points) Say whether the following are true or false:
   a. Gaussian elimination requires about twice as many arithmetic operations as Cholesky decomposition to solve $Ax = b$ when $A$ is symmetric positive-definite.
   Solution: True.
   b. Gaussian elimination applied to $Ax = b$ is stable without pivoting when $A$ is symmetric positive-definite.
   Solution: True.
   c. A matrix is ill-conditioned if its condition number is very small.
   Solution: False. A matrix is ill-conditioned if its condition number is very large.
   d. Suppose, for a particular problem, that the Gauss-Seidel iterates converge for every $x^{(0)}$. Then the iterates produced by SOR with the optimal relaxation parameter also converge for every $x^{(0)}$.
   Solution: True. The Gauss-Seidel iterates converge for every $x^{(0)}$ if and only if $\rho(T_{GJ}) < 1$. If $\omega_*$ is the optimal SOR relaxation parameter, then, by definition, $\rho(T_{\omega_*}) \leq \rho(T_{\omega})$ for all $\omega$. Since SOR reduces to Gauss-Seidel when $\omega = 1$, it follows in particular that $\rho(T_{\omega_*}) \leq \rho(T_{GJ}) < 1$; hence the SOR iterates with the optimal relaxation parameter also converge for ever $x^{(0)}$.
   e. Newton’s method with backtracking will always produce a solution from an arbitrary initial guess.
   Solution: False. The iterates can diverge on some problems. In fact, some nonlinear problems have no solutions, in which case the iterates can’t possibly converge.

2. (20 points) Consider $A = \begin{pmatrix} 1 & \delta - 1 \\ \delta - 1 & 1 \end{pmatrix}$ for $0 < \delta < 1$.
   a. (10 points) Compute the condition number $\kappa(A)$ as a function of $\delta$. You can use any norm you like, but indicate clearly which norm you are using.
   Solution: Using the 1-norm (the treatment with the $\infty$-norm is virtually identical), we have $\|A\|_1 = \max\{|1 + \delta - 1|, |\delta - 1| + 1\} = 2 - \delta$ for $0 < \delta < 1$. Also,
   \[
   A^{-1} = \frac{1}{1 - (\delta - 1)^2} \begin{pmatrix} 1 & 1 - \delta \\ 1 - \delta & 1 \end{pmatrix} = \frac{1}{\delta(2 - \delta)} \begin{pmatrix} 1 & 1 - \delta \\ 1 - \delta & 1 \end{pmatrix},
   \]
   so, for $0 < \delta < 1$,
   \[
   \|A^{-1}\|_1 = \frac{1}{\delta(2 - \delta)} \max\{|1 + |\delta - 1|, |\delta - 1| + 1\} = \frac{1}{\delta}.
   \]
   Then $\kappa(A) = (2 - \delta)/\delta$ for $0 < \delta < 1$. 


b. (10 points) Suppose $\delta = 10^{-6}$ and, for some $b$, you solve $Ax = b$ using Gaussian elimination with pivoting on a computer having eight decimal digits of accuracy in single precision and sixteen decimal digits of accuracy in double precision. According to the practical rule given in class, about how many trustworthy digits can you expect in the solutions computed in single and double precision?

Solution: For $\delta = 10^{-6}$, $\kappa(A) = (2 - 10^{-6})/10^{-6} \approx 2 \times 10^6 \approx 10^6$, so according to the practical rule, we can expect about $8 - 6 = 2$ trustworthy digits in single precision and about $16 - 6 = 10$ trustworthy digits in double precision.

3. (15 points) Suppose you want to determine the coefficients $x_1$ and $x_2$ in a linear model $y = x_1 + x_2t$, and you observe $(t, y)$-data $(-1, 3)$, $(0, 2)$, and $(1, 4)$. What are the coefficient matrix and the right-hand side of the normal equations in this case?

Solution: With $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, and $S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$, the normal equations are $S^T S x = S^T Y$. The coefficient and right-hand side are, respectively,

$S^T S = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $S^T Y = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$.

4. (15 points) For the matrix $A$ considered in problem 2, what can you say about the convergence of the Jacobi and Gauss–Seidel iterates? In particular, do they converge for all initial points? If the answer is “yes” for both methods, for which method are the iterates likely to converge faster? Give reasons for your answers. You may cite theorems on the handout, but be sure to say why they apply.

Solution: Since $1 > |\delta - 1| = 1 - \delta$ for $0 < \delta < 1$, $A$ is strictly diagonally dominant. It follows from Theorem 7.21 that $A$ is nonsingular, and both the Jacobi and Gauss–Seidel iterates converge to $A^{-1}b$ for every $x^{(0)}$. Since also $\delta - 1 < 0 < 1$ for $0 < \delta < 1$, we can apply the Stein–Rosenberg Theorem to obtain $0 < \rho(T_{GS}) = \rho(T_J) < 1$, and it follows that the Gauss–Seidel iterates are likely to converge faster than the Jacobi iterates. (With a little more work, one can show that the Gauss–Seidel iterates always converge faster than the Jacobi iterates.)

5. (20 points) Consider, for a given $x^{(0)}$, an iteration $x^{(k+1)} = Tx^{(k)} + c$ for $k = 0, 1, \ldots$, where $T = \begin{pmatrix} 1/2 & 1/3 \\ 0 & 1/4 \end{pmatrix}$ and $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

a. (5 points) What is the unique $x^*$ satisfying $x^* = Tx^* + c$?

Solution: We want $x^* = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 = \frac{1}{2}x_1 + \frac{1}{3}x_2 + 1$ and $x_2 = \frac{1}{4}x_2 + 0$. Solving these equations yields $x_2 = 0$ and $x_1 = 2$, so $x^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. 


b. (5 points) What is $\rho(T)$, the spectral radius of $T$?

Solution: Since $T$ is triangular, its eigenvalues are just its diagonal entries. Then $\rho(T) = \max\{|1/2|, |1/4|\} = 1/2$.

c. (10 points) Following the reasoning used in class, determine approximately how many iterations are required to reduce the error $\|x^{(k)} - x^*\|$ by a factor of 10. (You may assume that the norm is such that $\|T\| \approx \rho(T)$.)

Solution: We want $k$ such that $\rho(T)^k \leq 1/10$. This is equivalent to $k \log_{10}(\rho(T)) = k \log_{10}(1/2) = -k \log_{10}2 \leq -1$, which holds if and only if $k \geq 1/\log_{10}2 \approx 3.32$. So $k = 4$ will do the job.

6. (10 points) Write the two nonlinear equations
\[ e^{x_1} = \sin x_2 + 10 \quad \text{and} \quad x_1^2 = 1 - x_1 x_2 \]
in the vector system form $F(x) = 0$, and determine the Jacobian matrix $F'(x)$ for $F$.

Solution: There are various possible solutions. Here is perhaps the most straightforward one: With $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, set
\[ F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = \begin{pmatrix} e^{x_1} - \sin x_2 - 10 \\ x_1^2 - 1 + x_1 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0. \]

(The zero on the far right is, of course, the zero vector.) For this $F$, one has
\[ F'(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \frac{\partial F_1}{\partial x_2}(x) \\ \frac{\partial F_2}{\partial x_1}(x) & \frac{\partial F_2}{\partial x_2}(x) \end{pmatrix} = \begin{pmatrix} e^{x_1} & -\cos x_2 \\ 2x_1 + x_2 & x_1 \end{pmatrix}. \]