Solving Simple Systems of Linear Equations with Gaussian Elimination

The following illustrates the procedures and principles involved in solving simple systems of linear equations using Gaussian elimination followed by back substitution.

First, consider the linear system

$$\begin{aligned} x + 2y + z &= 5 \\ 2y + z &= 3 \\ 2z &= 2 \end{aligned} \tag{1}$$

This system is *upper-triangular*, meaning that the last equation involves only the last unknown, the next-to-last equation involves only the last two unknowns, and so on. We can very easily solve a triangular system by first solving the last equation for the last unknown, then plugging the value of the last unknown into the next-to-last equation and solving for the next-to-last unknown, and so on. This process is called *back substitution*. For the system (1), it yields $z = 1$, $y = \frac{1}{2}(3 - 1) = 1$, and $x = 5 - 2 \cdot 1 - 1 = 2$.

Now suppose we have a system that is not upper-triangular, say

$$\begin{aligned} x + 2y + z &= 5 \\ x + 4y + 2z &= 8 \\ x + 2z &= 4 \end{aligned} \tag{2}$$

We use *Gaussian elimination* to convert this to an upper-triangular system that has the same solution set. At the first step of Gaussian elimination, we subtract multiples of the first equation from the equations below to eliminate the first unknown from the equations below. At the second step, we subtract multiples of the (new) second equation from the equations below to eliminate the second variable (as well as the first) from the equations below. We continue in this way until we obtain a triangular system. For the system (2), which has three equations, there are only two steps of Gaussian elimination, as follows: first, subtract the first equation from the second and third; second, add the new second equation to the third. These are illustrated below.

$$\begin{aligned} x + 2y + z &= 5 \\ x + 4y + 2z &= 8 \\ x + 2z &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + 2y + z &= 5 \\ 2y + z &= 3 \\ -2y + z &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + 2y + z &= 5 \\ 2y + z &= 3 \\ 2z &= 2 \end{aligned} \tag{3}$$

We can recast this process in matrix terms using the *augmented matrix*, obtained by augmenting the coefficient matrix of the system with the right-hand side vector. For the system (2), this matrix is

$$\begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 1 & 4 & 2 & | & 8 \\ 1 & 0 & 2 & | & 4 \end{pmatrix} \tag{4}$$

In matrix terms, Gaussian elimination involves subtracting multiples of matrix rows from rows below until an upper-triangular matrix is obtained. These operations are called *elementary row operations*; for the system (2), these parallel the operations performed on the equations of the system in (3):

$$\begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 1 & 4 & 2 & | & 8 \\ 1 & 0 & 2 & | & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & -2 & 1 & | & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & 0 & 2 & | & 2 \end{pmatrix} \tag{5}$$

Viewing Gaussian elimination in matrix terms helps to bring out a very important point: *The steps of the elimination are determined by the coefficient matrix alone.* The right-hand side vector plays no role.

Now suppose we modify the third equation in (2) to obtain the system

$$\begin{aligned} x + 2y + z &= 5 \\ x + 4y + 2z &= 8 \\ 2y + z &= 3 \end{aligned} \tag{6}$$

Forming the augmented matrix and carrying out Gaussian elimination give

$$\begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 1 & 4 & 2 & | & 8 \\ 0 & 2 & 1 & | & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & 2 & 1 & | & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \tag{7}$$

The right-most matrix corresponds to the system

$$\begin{aligned} x + 2y + z &= 5 \\ 2y + z &= 3 \\ 0 &= 0, \end{aligned} \tag{8}$$

which has infinitely many solutions.

Now suppose we modify the right-hand side of the third equation in (6) to have

$$\begin{aligned} x + 2y + z &= 5 \\ x + 4y + 2z &= 8 \\ 2y + z &= 4 \end{aligned} \tag{9}$$

For this system, forming the augmented matrix and carrying out Gaussian elimination give

$$\begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 1 & 4 & 2 & | & 8 \\ 0 & 2 & 1 & | & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & 2 & 1 & | & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 2 & 1 & | & 3 \\ 0 & 0 & 0 & | & 1 \end{pmatrix} \tag{10}$$

The system corresponding to the right-most matrix is now

$$\begin{aligned} x + 2y + z &= 5 \\ 2y + z &= 3 \\ 0 &= 1, \end{aligned} \tag{11}$$

which has *no* solutions.

The "problem" with the systems (6) and (9) is that, in each case, Gaussian elimination applied to the augmented matrix results in a triangular matrix having a zero entry on the main diagonal. When this happens, the system does not have a unique solution but has either infinitely many solutions (as in (6)) or no solutions (as in (9)). This brings out a second very important point: *Whether a system has a unique solution is determined by the coefficient matrix alone. If the system does not have a unique solution, then it has either infinitely many solutions or no solution, depending on the right-hand side.*

Finally, consider the system

$$\begin{aligned} x + 2y + z &= 5 \\ x + 2y + 2z &= 8 \\ x + 2z &= 4 \end{aligned} \tag{12}$$

The first step of Gaussian elimination gives

$$\begin{aligned} x + 2y + z &= 5 \\ x + 2y + 2z &= 8 \\ x + 2z &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + 2y + z &= 5 \\ z &= 3 \\ -2y + z &= -1 \end{aligned} \tag{13}$$

or, in matrix terms,

$$\begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 1 & 2 & 2 & | & 8 \\ 1 & 0 & 2 & | & 4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 1 & | & 5 \\ 0 & 0 & 1 & | & 3 \\ 0 & -2 & 1 & | & -1 \end{pmatrix} \tag{14}$$

At this point, Gaussian elimination breaks down. Specifically, the second step of the algorithm calls for subtracting a multiple of the second equation from the third to eliminate the second unknown from the third equation. (In matrix terms, the second step calls for the subtracting a multiple of the second row from the third to make the second entry in the third row zero.) This cannot be done. The "problem" does not lie with the system; it has a unique solution that we could easily find with Gaussian elimination if we interchange the second and third equations. Rather, the "problem" is with our formulation of Gaussian elimination itself, which is too limited to see the need for and perform this interchanging operation.

From now on, we will refer to our current formulation of Gaussian elimination as *naive Gaussian elimination*. Our next order of business will be to outline simple "fixes" that allow the algorithm to proceed past such trivially resolved "problems" as those in (13)-(14).