Bounds for the electromagnetic material properties of a spatio-temporal dielectric polycrystal with respect to one-dimensional wave propagation

By Konstantin A. Lurie

Department of Mathematical Sciences, Worcester Polytechnic Institute, 100 Institute Road, Worcester MA 01609

Abstract

In this paper, we discuss the problem of attainability of various points of the hyperbola $\mathcal{E}/M = \varepsilon/\mu$ characterizing spatio-temporal dielectric polycrystals in one spatial dimension generated by an original isotropic dielectric with properties (ε, μ) . All points of this hyperbola appear to be attainable by spatio - temporal polycrystallic laminates.

Keywords: spatio-temporal polycrystals; subrelativistic polycrystals; relativistic (Cherenkov) polycrystals; wave impedance

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1. Introduction

As it has been shown before (Lurie 1998) for the electromagnetic wave propagation in one spatial dimension, if a dielectric mixture is assembled in space-time from isotropic dielectric constituents possessing the same ratio ε/μ of permittivity ε to permeability μ, then the ratio $\mathcal{E}/M$ of the effective permittivity $\mathcal{E}$ to the effective permeability M of the mixture will preserve the value ε/μ. This statement may be rephrased as the conservation law for the relevant wave impedance $\sqrt{M/\mathcal{E}}$; it holds true for any material pattern in (z, t)- plane if such a pattern allows for the existence of a solution of Maxwell's equations belonging to the relevant Sobolev space. Particularly, if we have a laminate in space-time assembled from the layers occupied by the same isotropic dielectric with material parameters ε, μ, and if this dielectric is brought to its own motion within each layer, then the laminate will become a *polycrystal* in space-time possessing the value $\mathcal{E}/M$ equal to ε/μ. In this paper, we investigate the problem of *attainability* of various portions of the hyperbola $\mathcal{E}/M = \varepsilon/\mu$ in the plane of effective parameters $\mathcal{E}c$, $1/Mc$, where c denotes the speed of light in the vacuum.

This problem is discussed below for spatio-temporal polycrystallic laminates in one spatial dimension.

Smooth solutions are possible for two types of such microstructures. For polycrystals of the first type, the speed of material motion within each fragment measured in a laboratory frame does not exceed the phase velocity $1/\sqrt{\varepsilon\mu}$ of waves in the relevant material; we shall call such polycrystals *subrelativistic* . For another admissible type of polycrystals (we shall call them *relativistic* or *Cherenkov* polycrystals), the material velocity in all fragments exceeds the phase velocity $1/\sqrt{\varepsilon\mu}$ in the material. For subrelativistic polycrystals, the following bound holds:

$$\infty > \mathcal{E}c + 1/Mc \geq \varepsilon c + 1/\mu c; \tag{1}$$

both parameters $\mathcal{E}$ and M are then positive, they belong to the part of the hyperbola $\mathcal{E}/M = \varepsilon/\mu$ starting at the point $\varepsilon c, 1/\mu c$ $(\varepsilon c > 1/\mu c)$, and spreading towards the asymptote $\mathcal{E}c \to \infty, 1/Mc \to 0$. This particularly means that the point on the diagonal where $\mathcal{E}c = 1/Mc = \sqrt{\varepsilon/\mu}$, *cannot* be attained by a subrelativistic polycrystal.

For relativistic polycrystals, there holds a different bound

$$2\sqrt{\varepsilon/\mu} < \mathcal{E}c + 1/Mc \leq \varepsilon c + 1/\mu c, \tag{2}$$

which means that the point $\mathcal{E}c = 1/Mc = \sqrt{\varepsilon/\mu}$ *may* be approached arbitrarily closely by such a polycrystal.

2. Rank one laminates in space-time: general formulae

The effective parameters of such laminates have been computed in (Lurie 1997) under the assumption that the (periodic) laminate is assembled from two isotropic materials characterized by different pairs of values of $\varepsilon = \varepsilon(z,t)$ and $\mu = \mu(z,t)$:

$$(\varepsilon(z,t), \mu(z,t)) = \begin{cases} (\varepsilon_1, \mu_1) - \text{material 1} \\ (\varepsilon_2, \mu_2) - \text{material 2} \end{cases} \tag{3}$$

The materials were assumed immovable with respect to a laboratory frame and placed within alternating layers in $(z,t)-$ plane, these layers occupying, respectively, the mth and $(1-m)$th part of the period of the microstructure. The slope $V = dz/dt$ of the layers (the velocity of the material pattern) was so chosen as to ensure the observance of inequality

$$\frac{V^2 - c_1^2}{V^2 - c_2^2} \geq 0, \tag{4}$$

where $c_i = 1/\sqrt{\varepsilon_i\mu_i}, i = 1,2$, is the phase velocity of waves in ith material. This inequality is necessary to guarantee smoothness of the relevant solution (the absence of shocks).

It has been shown in (Lurie 1997,1998) that the system

$$\varepsilon u_t = v_z, \quad \frac{1}{\mu} u_z = v_t, \tag{5}$$

governing the electromagnetic field

$$\mathbf{E} = u_t \mathbf{j}, \quad \mathbf{B} = u_z \mathbf{i}, \quad \mathbf{H} = v_t \mathbf{i}, \quad \mathbf{D} = v_z \mathbf{j} \tag{6}$$

within original isotropic dielectrics, will, after homogenization, be replaced by the system

$$\begin{aligned} \alpha c u_z + \beta u_t &= V v_z + v_t, \\ V u_z + u_t &= \theta(\alpha c v_z + \beta v_t), \end{aligned} \tag{7}$$

with parameters α, β, θ defined as

$$\begin{aligned} \alpha &= \frac{1}{c} \frac{\left\langle \frac{1}{\varepsilon\mu\left(V^2 - c_i^2\right)} \right\rangle}{\left\langle \frac{1}{\varepsilon\left(V^2 - c_i^2\right)} \right\rangle}, \quad \beta = V \frac{\left\langle \frac{1}{V^2 - c_i^2} \right\rangle}{\left\langle \frac{1}{\varepsilon(V^2 - c_i^2)} \right\rangle}, \\ \theta &= \frac{\left\langle \frac{1}{\varepsilon(V^2 - c_i^2)} \right\rangle}{\left\langle \frac{1}{\mu(V^2 - c_i^2)} \right\rangle}, \end{aligned} \tag{8}$$

where

$$\langle \cdot \rangle = m_1 (\cdot)_1 + m_2 (\cdot)_2.$$

In Eqs. (2.5), we preserved original symbols u, v to denote the weak limits of the relevant quantities, i.e. their values averaged over the cell of periodicity. An equivalent form of Eqs. (2.5) is given by (Lurie 1998)

$$p u_z - q u_t = v_t, \quad q u_z + r u_t = v_z, \tag{9}$$

with parameters p, q, r defined as

$$p = \frac{V^2 - \theta\alpha^2 c^2}{\theta(\beta V - \alpha c)}, \quad q = -\frac{V - \theta\alpha c\beta}{\theta(\beta V - \alpha c)}, \quad r = -\frac{1 - \theta\beta^2}{\theta(\beta V - \alpha c)}. \tag{10}$$

Introduce the "primed" coordinate frame z', t' moving with velocity w with respect to the laboratory frame z, t. Coordinates z', t' are linked with z, t by the Lorentz formulae

$$z' = \gamma^{-1}(z - wt), \quad t' = \gamma^{-1}\left(t - \frac{w}{c^2} z\right), \quad \gamma = \sqrt{1 - w^2/c^2}. \tag{11}$$

If w is defined as the root of

$$\frac{q}{c^2}w^2 + \left(\frac{p}{c^2} - r\right) w + q = 0, \tag{12}$$

then the system (2.7) is reduced to the equations

$$(p + qw)u_{z'} = v_{t'}, \quad \left(\frac{p}{c^2} + \frac{q}{w}\right) u_{t'} = v_{z'}, \tag{13}$$

specifying the effective parameters $\mathcal{E}c$, $1/Mc$ (eigenvalues of the material tensor of a composite) through the formulae (cf. (2.3))

$$\mathcal{E}c = \frac{p}{c} + \frac{qc}{w}, \quad \frac{1}{Mc} = \frac{p}{c} + \frac{qw}{c}. \tag{14}$$

Applying direct inspection and referring to (2.10), we obtain the following expression for the second invariant of a material tensor:

$$\frac{\mathcal{E}}{M} = \left(\frac{p}{c} + \frac{qc}{w}\right)\left(\frac{p}{c} + \frac{qw}{c}\right) = pr + q^2,$$

and Eqs. (2.8) yield

$$\frac{\mathcal{E}}{M} = pr + q^2 = 1/\theta. \tag{15}$$

We will also need the formula for the first invariant $\mathcal{E}c + 1/M$ of the effective tensor of material parameters. This formula follows from (2.12) and (2.10):

$$\mathcal{E}c + 1/Mc = \frac{2p}{c} + \frac{qc}{w} + \frac{qw}{c} = \frac{p}{c} + rc. \tag{16}$$

Given Eqs. (2.8), we rewrite (2.14) in the form

$$\mathcal{E}c + 1/Mc = \frac{1}{\beta(V/c) - \alpha}\left[\left(\frac{V^2}{c^2} - 1\right)\frac{1}{\theta} - \left(\alpha^2 - \beta^2\right)\right]. \tag{17}$$

Eqs. (2.5)-(2.15) also hold for an arbitrary number n of layers in the cell of periodicity, these layers occupied each by its own dielectric with material parameters (ε_i, μ_i), $i = 1, ..., n$. The operation $\langle\cdot\rangle$ will then be defined as $\langle\cdot\rangle = \sum_{i=1}^{n} m_i (\cdot)_i$ where $m_i \geq 0$, $\sum_{i=1}^{n} m_i = 1$ represent the relevant volume fractions.

In Appendix, we establish a similar set of formulae, applicable, however, in a more general context. Specifically, we assume that the dielectric materials within the layers in a laminate may be brought each to its own motion, with the respective velocities $v_i, i = 1, 2..., n$, specified with regard to a laboratory frame. The discontinuous velocity pattern may be implemented through the use of the following feasible construction. Assume that we have a linear arrangement of caterpillars placed one after another along the $z-$ axis and carrying the dielectric tracks (figure 1). The tracks moved by caterpillars become electrically connected when they belong to the $z-$ axis, and stay disconnected otherwise. The $z-$ axis will then become occupied by material fragments moving each at its own horizontal velocity, and the electric current will flow along the $z-$ axis through the assemblage of electrically connected tracks. With this construction, the performance of the electromagnetic field will be controlled directly by an appropriate specification of the velocity pattern.

Under these assumptions, Eqs. (2.5) and (2.7)-(2.15) continue to hold, with Eqs. (2.6) replaced by

$$\begin{aligned} \alpha &= \frac{\left\langle \frac{A-Gth\psi}{\Delta} \right\rangle}{\left\langle \frac{1}{\Delta} \right\rangle}, \quad \beta = \frac{\left\langle \frac{G-Cth\psi}{\Delta} \right\rangle}{\left\langle \frac{1}{\Delta} \right\rangle}, \\ \theta &= \frac{\left\langle \frac{1}{\Delta} \right\rangle}{\left\langle \frac{\varepsilon/\mu}{\Delta} \right\rangle}, \end{aligned} \tag{18}$$

and parameters $\psi, A, G, C, \Delta, \varphi$ defined as

$$\begin{aligned} th\psi &= V/c, \\ A &= \frac{1}{\mu c} ch^2\varphi - \varepsilon c\, sh^2\varphi, \quad G = \left(\frac{1}{\mu c} - \varepsilon c \right) sh\varphi ch\varphi, \\ C &= \frac{1}{\mu c} sh^2\varphi - \varepsilon c\, ch^2\varphi, \quad \Delta = -A + 2Gth\psi - Cth^2\psi, \\ th\varphi &= v/c. \end{aligned} \tag{19}$$

The reader will not be confused with the identity of the symbol v used for material velocity in (2.17) and that used for the function v first appeared in (2.3). The real meaning of this symbol will become clear from the context in each individual case.

Note the identities

$$A = \varepsilon c + 1/\mu c + C, \quad G^2 - AC = \varepsilon/\mu. \tag{20}$$

We apply $\varepsilon = \varepsilon_i$, $\mu = \mu_i$, $v = v_i$ (or $\varphi = \varphi_i$) for the ith fragment (layer) in the lamination. When all φ_i become equal to zero (immovable materials), Eqs. (2.16) reduce to (2.6).

At another extreme, the materials may differ only in their values of φ_i alone; from the viewpoint of the electromagnetic properties, we may speak of the set of fragments of the same material (ε, μ) , with material in each fragment brought to its own motion with velocity $v_i = c\, th\varphi_i$. The relevant assemblage may be qualified as the spatio - temporal polycrystallic laminate, and the bounds (1.1), (1.2) will now be confirmed for this special microstructure.

3. Spatio-temporal polycrystallic laminates: the bounds

Consider the expression (2.15) for the first invariant $\mathcal{E}c+1/Mc$ in this case. Given Eqs. (2.16)-(2.18), we arrive at the following relations:

$$\begin{aligned}\theta &= \mu/\varepsilon,\\ \beta(V/c)-\alpha &= \frac{1}{\left\langle\frac{1}{\Delta}\right\rangle}\left[\frac{1}{\Delta}\left(Gth\psi - Cth^2\psi - A + Gth\psi\right)\right] = \frac{1}{\left\langle\frac{1}{\Delta}\right\rangle},\end{aligned}$$

$$\mathcal{E}c+1/Mc = \left(th^2\psi-1\right)\frac{\varepsilon}{\mu}\left\langle\frac{1}{\Delta}\right\rangle - \frac{\left\langle\frac{A-Gth\psi}{\Delta}\right\rangle^2}{\left\langle\frac{1}{\Delta}\right\rangle} + \frac{\left\langle\frac{G-Cth\psi}{\Delta}\right\rangle^2}{\left\langle\frac{1}{\Delta}\right\rangle}. \tag{21}$$

Because, by (2.18),

$$\frac{\varepsilon}{\mu}\left\langle\frac{1}{\Delta}\right\rangle = \left\langle\frac{G^2-AC}{\Delta}\right\rangle,$$

we rewrite the first term at the rhs of (3.1) as

$$\begin{aligned}\left(th^2\psi-1\right)\frac{\varepsilon}{\mu}\left\langle\frac{1}{\Delta}\right\rangle &= th^2\psi\left\langle\frac{G^2-AC}{\Delta}\right\rangle - \left\langle\frac{G^2+C^2th^2\psi-2GCth\psi}{\Delta}\right\rangle +\\ &+\left\langle\frac{AC+C^2th^2\psi-2GCth\psi}{\Delta}\right\rangle = th^2\psi\left\langle\frac{G^2-AC}{\Delta}\right\rangle - \left\langle\frac{(G-Cth\psi)^2}{\Delta}\right\rangle -\\ &-\langle C\rangle = th^2\psi\left\langle\frac{G^2-AC}{\Delta}\right\rangle - \left\langle\frac{(G-Cth\psi)^2}{\Delta}\right\rangle + \varepsilon c + 1/\mu c - \langle A\rangle =\\ &= \varepsilon c+1/\mu c - \left\langle\frac{(G-Cth\psi)^2}{\Delta}\right\rangle + \left\langle\frac{(A-Gth\psi)^2}{\Delta}\right\rangle.\end{aligned}$$

The expression (3.1) for $\mathcal{E}c+1/Mc$ now obtains the form

$$\begin{aligned}\mathcal{E}c+1/Mc &= \varepsilon c+1/\mu c + \left\langle\frac{(A-Gth\psi)^2}{\Delta}\right\rangle - \frac{\left\langle\frac{A-Gth\psi}{\Delta}\right\rangle^2}{\left\langle\frac{1}{\Delta}\right\rangle} -\\ &\quad - \left[\left\langle\frac{(G-Cth\psi)^2}{\Delta}\right\rangle - \frac{\left\langle\frac{G-Cth\psi}{\Delta}\right\rangle^2}{\left\langle\frac{1}{\Delta}\right\rangle}\right].\end{aligned} \tag{22}$$

Consider the quantity

$$F(x) = \left\langle\frac{x^2}{\Delta}\right\rangle - \frac{\left\langle\frac{x}{\Delta}\right\rangle^2}{\left\langle\frac{1}{\Delta}\right\rangle},$$

where symbols x and Δ are defined for each of the n materials (fragments). For the sake of simplicity, assume that $n = 2$; then it is easy to show that

$$F(x) = \frac{m_1 m_2}{\overline{\Delta}}(\Delta x)^2,$$

where $\overline{\Delta} = m_1\Delta_2 + m_2\Delta_1$, $\Delta x = x_2 - x_1$.

Eq. (3.2) is now rewritten as

$$\mathcal{E}c + 1/Mc = \varepsilon c + 1/\mu c + \frac{m_1 m_2}{\overline{\Delta}}\left[(\Delta A - \Delta G th\psi)^2 - (\Delta G - \Delta C th\psi)^2\right]. \tag{23}$$

For the case of a polycrystal,

$$\Delta A = (1/\mu c)\Delta(ch^2\varphi) - \varepsilon c\Delta\left(sh^2\varphi\right), \quad etc.$$

Since $\Delta C = \Delta A$ because of (2.18), we rewrite the second term in the rhs of (3.3) as

$$\frac{m_1 m_2}{\overline{\Delta}}\left[(\Delta A)^2 - (\Delta G)^2\right]\left(1 - th^2\psi\right). \tag{24}$$

Introduce the angle χ by $th^2\chi = 1/\varepsilon\mu c^2$; then

$$\Delta = -A + 2G th\psi - C th^2\psi =$$
$$-\varepsilon c\, ch^2\varphi\left[th^2\chi - th^2\varphi + 2\left(-th^2\chi + 1\right)th\varphi th\psi + \left(th^2\chi th^2\varphi - 1\right)th^2\psi\right] =$$
$$-\varepsilon c\, ch^2\varphi\,(1 - th\varphi th\psi)^2\left[th^2\chi - th^2(\varphi - \psi)\right],$$

and the expression (3.4) becomes equal to

$$-\frac{m_1 m_2(1 - th^2\psi)}{\varepsilon c\overline{D}}\left[(\Delta A)^2 - (\Delta G)^2\right], \tag{25}$$

$$\overline{D} = m_1 D_2 + m_2 D_1;\ D_i = -\frac{1}{\varepsilon c}\Delta_i = ch^2\varphi_i(1 - th\varphi_i th\psi)^2\left[th^2\chi - th^2(\varphi_i - \psi)\right],\ i = 1, 2. \tag{26}$$

Referring to (2.17), we calculate the difference $(\Delta A)^2 - (\Delta G)^2$ as

$$\begin{aligned}&(\varepsilon c - 1/\mu c)^2\left[-sh^2\varphi_2 ch^2\varphi_2 + 2sh\varphi_2 ch\varphi_2 sh\varphi_1 ch\varphi_1 - sh^2\varphi_1 ch^2\varphi_1 -\right.\\ &\left.+\left(ch^4\varphi_2 - 2ch^2\varphi_2 ch^2\varphi_1 + ch^4\varphi_1\right)\right] = -(\varepsilon c - 1/\mu c)^2 sh^2(\varphi_1 - \varphi_2).\end{aligned} \tag{27}$$

Getting back to (3.3) and using (3.4)-(3.7) along with the definition $th\chi = 1/c\sqrt{\varepsilon\mu}$ of χ, we conclude that

$$\mathcal{E}c + 1/Mc = \varepsilon c + 1/\mu c + k, \tag{28}$$

where

$$k = \frac{\varepsilon c}{\overline{D}} m_1 m_2\left(1 - th^2\psi\right)\left(1 - th^2\chi\right)^2 sh^2(\varphi_1 - \varphi_2).$$

We now consider two admissible cases listed above.

(i) *Subrelativistic case.*

In this case, $\chi \geq \varphi_i - \psi$, and $D_i \geq 0, \quad i = 1,2$. Because $th\psi = V/c \leq 1$, we see that $k \geq 0$, and the right Ineq. (1.1) is confirmed. To confirm the left one, find the maximum of k as the function of m_1 :

$$\max_{m_1} k = \varepsilon c \left(1 - th^2\psi\right) \left(1 - th^2\chi\right)^2 sh^2 \left(\varphi_1 - \varphi_2\right) \frac{1}{\left(\sqrt{D_1} + \sqrt{D_2}\right)^2}; \qquad (29)$$

the maximizing value of m_1 is defined as

$$m_1 = \frac{\sqrt{D_1/D_2}}{1 + \sqrt{D_1/D_2}}.$$

Consider the limit value of $\max_{m_1} k$ attained as $\varphi_1 = \psi$, $\varphi_2 \to \chi + \psi = \chi + \varphi_1$; then $D_2 \to 0$, and we get

$$\lim \max_{m_1} k = \varepsilon c(1 - th^2\chi) = \varepsilon c - 1/\mu c, \qquad (30)$$

and the relevant value $\lim m_1 = 1$. This result is not paradoxical since material 2 is then disappearing more slowly than the value D_2 tends to zero (observe that we are considering the value of m_1 *maximizing* k all the time!) If we first go to $m_1 = 1$, and *then apply* the limit $D_2 \to 0$, then the limit value of k would become zero indicating that we *first* withdraw material 2.

Given Eq. (3.10) and referring to (3.8) and to the conservation law $\mathcal{E}/M = \varepsilon/\mu$, we conclude that the original point $P\left(\varepsilon c, 1/\mu c\right)$ on the hyperbola (figure 2) is now replaced by the point P_1 with coordinates $\mathcal{E}_1 c = \varepsilon c \left(1 + \sqrt{1 - th^2\chi}\right)$, $\frac{1}{M_1 c} = \frac{1}{\mu c} \frac{1}{1+\sqrt{1-th^2\chi}}$; with this point we associate a new angle χ_1 defined by

$$th^2\chi_1 = \frac{1}{c^2\mathcal{E}_1 M_1} = \frac{1}{c^2\varepsilon\mu\left(1 + \sqrt{1 - th^2\chi}\right)^2} = \frac{th^2\chi}{\left(1 + \sqrt{1 - th^2\chi}\right)^2} = \frac{1 - \sqrt{1 - th^2\chi}}{1 + \sqrt{1 - th^2\chi}},$$

and

$$1 - th^2\chi_1 = \frac{2\sqrt{1 - th^2\chi}}{1 + \sqrt{1 - th^2\chi}} \geq \sqrt{1 - th^2\chi}.$$

Repeating this procedure with P_1 as a starting point, we arrive after this next step at a new point P_2 with coordinates

$$\mathcal{E}_2 c = \mathcal{E}_1 c \left(1 + \sqrt{1 - th^2\chi_1}\right) \geq \varepsilon c \left[1 + \left(1 - th^2\chi\right)^{1/2}\right] \left[1 + \left(1 - th^2\chi\right)^{1/4}\right],$$

$$1/M_2 c = (1/M_1 c)\left(1+\sqrt{1-th^2\chi_1}\right)^{-1} \le$$
$$\le \frac{1}{\mu c}\left[1+\left(1-th^2\chi\right)^{1/2}\right]^{-1}\left[1+\left(1-th^2\chi\right)^{1/4}\right]^{-1},$$

and so on. Because the infinite product

$$(1+x)\left(1+x^{1/2}\right)\left(1+x^{1/4}\right).....,\quad 0<x\le 1$$

is divergent, we manage to cover the whole branch $\mathcal{E} \ge \varepsilon c$, $1/Mc \le 1/\mu c$ of the hyperbola $\mathcal{E}/M = \varepsilon/\mu$, and the right Ineq. (1.1) becomes confirmed.

Remark 3.1. The point P_2 in the above construction corresponds to the rank two polycrystallic laminate, etc; we thus apply *laminates of multiple rank* to prove attainability of the relevant part of the hyperbola.

Remark 3.2. The successive procedure just described fails to work once $\chi = \infty$, i.e. once the original material represents the vacuum.

(ii) *Relativistic (Cherenkov) case.*

For this case, $\chi \le \varphi_i - \psi$, and all D_i become negative. The function k attains its minimum with respect to m_1, this minimum being defined by the formula

$$\min_{m_1} k = -\varepsilon c\left(1-th^2\psi\right)\left(1-th^2\chi\right)^2 sh^2(\varphi_1-\varphi_2)\frac{1}{\left(\sqrt{|D_1|}+\sqrt{|D_2|}\right)^2},$$

similar to (3.9).

A simple calculation shows that the function $\min_{m_1} k$ of the argument φ_1, φ_2 attains its minimum when $\varphi_1 = \psi+\chi, \varphi_2 = \infty$, or when $\varphi_1 = \infty, \varphi_2 = \psi+\chi$. Assume that $\varphi_2 \to \infty$ (i.e. material 2 moves at speed *c); then*

$$\begin{aligned}\lim_{\varphi_2\to\infty}\min_{m_1} k &= -\varepsilon c\frac{1+th\psi}{1-th\psi}\left(1-th^2\chi\right)\lim_{\varphi_2\to\infty}\frac{sh^2(\varphi_1-\varphi_2)}{ch^2\varphi_2}\\ &= -\varepsilon c e^{2(\psi-\varphi_1)}\left(1-th^2\chi\right).\end{aligned}$$

Because $\chi \le \varphi_1 - \psi$ in this case, we obtain the attainable lower bound by taking $\varphi_1 = \chi + \psi$:

$$k \ge \min_{m_1} k \ge -\varepsilon c e^{-2\chi}\left(1-th^2\chi\right) = -\varepsilon c\left(1-th\chi\right)^2 = -\left(\sqrt{\varepsilon c}-\frac{1}{\sqrt{\mu c}}\right)^2.$$

Getting back to (3.8) we conclude that in the relativistic case

$$\mathcal{E}c + 1/Mc \geq \varepsilon c + 1/\mu c - \left(\sqrt{\varepsilon c} - \frac{1}{\sqrt{\mu c}}\right)^2 = 2\sqrt{\varepsilon/\mu}.$$

Together with $k \leq 0$, this confirms Ineqs. (1.2). Particularly, the point $\mathcal{E}c = 1/Mc = \sqrt{\varepsilon/\mu}$ formally becomes attainable in this case (see Remark 3.3 below).

The above argument can be reformulated to include polycrystals assembled from more than two moving fragments.

Remark 3.3. The angle ψ does not actually affect the analysis of the Cherenkov case whereas it appears to be substantial in the subrelativistic case.

The limit $\varphi_2 = \infty$ cannot be attained for particles with non-zero proper mass because the procedure requires material motion at the velocity v equal to c. Also, when $v \to c$, then the wavelength λ of original disturbance u measured in the frame moving at the speed v, tends to zero as $v \to c$, and this violates the assumption $d/\lambda \ll 1$ where d is the characteristic length of lamination. For both reasons, the attainability of the point $\mathcal{E}c = 1/Mc = \sqrt{\varepsilon/\mu}$ appears to be only formal. But this point may be physically approached as close as desired by taking φ_2 sufficiently large.

Corollary 3.1. Given two materials with properties (ε_1, μ_1) and (ε_2, μ_2), $\varepsilon_2/\mu_2 > \varepsilon_1/\mu_1$, we may assemble laminates with effective properties $\mathcal{E}c$, $1/Mc$ occupying the entire strip $\mathcal{E}c > 1/Mc$, $\varepsilon_2/\mu_2 > \mathcal{E}/M > \varepsilon_1/\mu_1$. The latter inequality follows from Eq. (2.13) and the last Eq. (2.16) that hold for a general laminate, if we apply these equations to a laminate assembled from layers occupied by the two materials and observe that the values of Δ should have the same sign in each layer.

Corollary 3.2. Consider the set LU of *all possible* laminates assembled in space-time from the elements of an *arbitrary set* U of original constituents (ε_s, μ_s). Let (ε_1, μ_1) and (ε_2, μ_2) be the *extreme* points of U, i.e. the elements satisfying the inequalities

$$\varepsilon_2/\mu_2 > \varepsilon_s/\mu_s > \varepsilon_1/\mu_1$$

for all s. Then, for every laminate $(\mathcal{E}, M)$ in LU

$$\varepsilon_2/\mu_2 > \mathcal{E}/M > \varepsilon_1/\mu_1.$$

In other words, the set LU is entirely specified by its extreme elements (ε_1, μ_1) and (ε_2, μ_2).

Appendix A

If the plane electromagnetic waves propagate along the $z-$ axis, then the electromagnetic field (2.4) is characterized by two electromagnetic tensors (Lurie 1998)

$$F = \sqrt{2}c\left(u_{x_3}a_{23} + u_{x_4}a_{24}\right), \tag{A1}$$

$$f = \sqrt{2}ic\left(v_{x_4}a_{23} - v_{x_3}a_{24}\right), \tag{A2}$$

constructed from the vectors $\mathbf{B}, \mathbf{E}, \mathbf{H},\ \mathbf{D}$ taken pairwise. Here and below, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ denote the Minkowskian coordinates, and - u, v denote, respectively, the x_2 - component of the electric vector potential $\mathbf{A}$ and the x_1 - component of the magnetic vector potential $\mathbf{A}^*$. The antisymmetric tensors

$$a_{23} = \left(1/\sqrt{2}\right)\left(i_2 i_3 - i_3 i_2\right), \quad a_{24} = \left(1/\sqrt{2}\right)\left(i_2 i_4 - i_4 i_2\right)$$

are constructed with the aid of the unit vectors i_1, i_2, i_3, i_4 generating an orthonormal basis in Minkowskian space. These tensors satisfy the orthonormality conditions

$$a_{23} : a_{24} = 0, \ \ a_{23} : a_{23} = a_{24} : a_{24} = -1.$$

The tensors F and f are linked through the material relation

$$f = s : F, \tag{A3}$$

where the fourth rank material tensor s is given for an immovable material by the formula

$$s = -\frac{1}{\mu c}a_{23}a_{23} - \varepsilon c\, a_{24}a_{24}. \tag{A4}$$

If the dielectric is brought into motion with a uniform speed v along the x_3- axis, then the relevant expression for s becomes

$$s = -\frac{1}{\mu c}a'_{23}a'_{23} - \varepsilon c\, a'_{24}a'_{24}, \tag{A5}$$

with the "primed" tensors a'_{23}, a'_{24} defined as

$$a'_{23} = a_{23}ch\varphi + ia_{24}sh\varphi, \quad a'_{24} = -ia_{23}sh\varphi + a_{24}ch\varphi, \tag{A6}$$

and the angle φ defined by $th\varphi = v/c$. Referring to (A5) and (A6), we reduce the tensor relation (A3) to the system of two equations

$$\begin{aligned} Au_{x_3} + i\,Gu_{x_4} &= iv_{x_4}, \\ -Gu_{x_3} - i\,Cu_{x_4} &= v_{x_3}, \end{aligned} \tag{A7}$$

with parameters A, G, C defined by (2.17).

Consider now two dielectric media moving with different speeds v_1 and v_2 along the x_3- axis, and let these media be separated by a point moving with velocity $V \leq c$ along the same axis. The world line L of this point will become a straight line with the slope V in (z,t)-plane.

The derivative u_τ of u along this world line will be computed as $(th\psi = V/c)$

$$u_\tau = iu_{x_3} th\psi - u_{x_4}; \tag{A8}$$

this derivative should be continuous across L along with a similar derivative of v (Sommerfeld 1964). Bearing this in mind, we eliminate u_{x_4}, v_{x_4} from (A7) and arrive, after some calculation, at the system

$$\begin{aligned} u_{x_3} &= iu_\tau \frac{Cth\psi - G}{\Delta} + iv_\tau \frac{1}{\Delta}, \\ v_{x_3} &= iu_\tau \frac{G^2 - AC}{\Delta} + iv_\tau \frac{Cth\psi - G}{\Delta}, \end{aligned}$$

with Δ defined by (2.17).

We now take average values of both sides of either equation bearing in mind the continuity of u_τ, v_τ. As we go back to notation (A8) after averaging, we arrive, after some calculation, at Eqs. (2.5) with parameters α, β, θ defined by Eqs. (2.16). As noted before, we preserve the original symbols u, v in Eqs. (2.5) to denote the weak limits of the relevant quantities.

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References

Lurie, K.A. 1998 The problem of effective parameters of a mixture of two isotropic dielectrics distributed in space-time and the conservation law for wave impedance in one-dimensional wave propagation. *Proc. R. Soc. Lond.* A **454**, 1767-1779.

Lurie, K.A. 1997 Effective properties of smart elastic laminates and the screening phenomenon. *Int. J. Solids Structures* **34,** 1633-1643.

Sommerfeld, A. 1964 *Elektrodynamik.* Leipzig: Geest & Portig.

Legend

Figure 1. The caterpillar construction.
Figure 2. The hyperbola $\mathcal{E}/M = \varepsilon/\mu$.