On homogenization of activated laminates in 1D-space and time

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Dedicated to Professor Vladimir A. Palmov on the occasion of his 75th birthday.

In this paper, I discuss observations indicating that homogenization, in its standard version based on weak compactness, generally does not work for dynamic materials with arbitrary material geometry. A remarkable exception is represented by the spatial-temporal laminates where such compactness can be established. For the slow (timelike) laminates, this was done in [18] with the use of energy bound. In the present work, a similar result is established for the fast (spacelike) laminates, this time with the use of momentum flux bound.

1 Introduction

Heterogeneous material structures assembled in space and time – dynamic materials (DM) – have received a systematic investigation starting from the late nineties [1, 2] when it was realized that such formations generate a novel material concept. An idea of spatial-temporal variability of material properties may seem unusual to a conservative mind. Though this principle was being implemented for ages in living organisms, its technological implementation has become achievable relatively recently. At the same time, viewed as a material concept, this idea goes far and wide: it embraces phenomena that are at first sight very diverse. Among them, one can mention the traffic flow, the picture dynamics observed on a television screen, the patterns of material properties in mechanical devices, specifically on a nanoscale [3, 4]. An attentive observer will find indications of this concept in mechanics of bodies of variable mass, specifically, in the rocket flight; many illustrations also come form vibrational mechanics [5]. A classical example that “lies on the surface” is set by electrodynamics of moving media: the material relations introduced by Minkowski are explicitly affected by the material motion. Generally speaking, electrodynamics offers an ideal embodiment for the concept of dynamic materials. The tensor formulation of electrodynamics has revealed a natural classification of dynamic materials, such as dielectrics and conductors, into activated and kinetic [1, 2]. Though it originally appeared in the context of electrodynamics [6], such classification is quite universal. The activated DM are defined as substances that expose spatio-temporal variability of their property patterns alone, with no presence of actual material motion. The kinetic DM are specified as assemblages of material fragments that participate in relative mechanical motion: the Minkowski’s material relations clearly illustrate this possibility.

Various means can be used to maintain the material property patterns variable in space and time. In electromagnetic applications, such variability can be achieved by the network switching artificially implemented in both radio and optical frequency ranges [7, 8]. In the traffic flow, it may be realized by the use of traditional signals, such as traffic lights and road signs, applied wherever and whenever necessary. Regardless of the implementation, all of the DM possess a common feature that is quite universal: such materials appear to be thermodynamically open systems. To secure their existence, a non-stop exchange of energy and momentum should be maintained between the material formation and the environment. In other words, an external agent should necessarily be involved as a factor that supports a spatio-temporal material structure.

The role played by DM in material design comes from their ability to effectively control dynamic processes, especially, the wave propagation. With DM, waves can be averted from entering undesirable locations, their energy can be dramatically increased by pumping it in from outside, or it can be released and utilized for purpose. The critical frequency in waveguides can be controlled and even removed by the DM fillings, the negative index electromagnetic materials may be assembled, to name just a few important effects. In the context of design, special mention should be made about the DM built from conventional materials on a spatio-temporal microscale. This class of DM is naturally termed dynamic composites (DC).
Composites assembled in space alone have become a commonplace in material design. In a static environment, the involvement of composites is almost inevitable when a design becomes optimal because the requirement of optimality can generally be satisfied due to their appearance [9, 10]. Mathematically, the composites enter the optimal design in statics through the procedure of homogenization that specifies the effective properties of composites, i.e., the properties demonstrated on a scale much larger than the characteristic scale of material inhomogeneities. Homogenization is then implemented in the context of elliptic equations, and this procedure can theoretically be conducted for an extremely wide class of microgeometries, though the explicit formulae for effective properties are available only for laminates. With regard to the differential operator div u grad, it has been shown in [11] that laminates of higher rank may replace an arbitrary microstructure as formations possessing the same effective material constants.

It is alluring to take a similar approach in dynamics with DM appearing as a material medium that supports the wave propagation. The waves characterized by a spatio-temporal scale much larger than the scale of material distribution may be expected to travel through the DM perceiving it as a homogeneous substance characterized by some effective material properties. This idea has received a successful analytic implementation with respect to laminate structures in space and time [1, 12]; as a result, the explicit formulae were obtained for effective material parameters. However, the attempts to apply homogenization toward other microgeometries encountered difficulties that are more serious than purely technical. The analysis of such difficulties revealed a peculiar position occupied by laminates among the DM.

In the following section we give a brief account of reasons that make homogenization of DM a special problem.

2 Specifics of homogenization applied to the wave equation

Consider the wave equation
\[ (\rho u)_t - (ku_z)_z = 0, \] (1)
with coefficients \( \rho, k \) dependent on \( z, t \). This equation is equivalent to the system
\[ v_t = ku_z, \quad v_z = \rho u_t. \] (2)
The coefficients will be assumed taking at each point \((z, t)\) one out of two pairs of values: \( \rho_1, k_1 \) (“material 1”), or \( \rho_2, k_2 \) (“material 2”); along with the pairs \( \rho_i, k_i (i = 1, 2) \) of material parameters, we will also refer to an equivalent pair \( \gamma_i = \sqrt{k_i/\rho_i}, \quad a_i = \sqrt{k_i/\rho_i} \), denoting, respectively, the wave impedance and the phase velocity for each material. Without loss of generality, we will assume that \( a_2 > a_1 \). The distribution of materials in a \((z, t)\)-plane affects the wave propagation. We will be looking for smooth solutions to (1) or (2), i.e., solutions \( u, v \) that are continuous across any regular interface \( \Gamma \) separating zones occupied by different materials. Such solutions may exist if the slope \( w = dz/dt \) of the interface satisfies either one of the two inequalities:

\[ |w| < a_1, \text{ or } |w| > a_2. \] (3)

This requirement means that there is precisely one pair of characteristics arriving at \( \Gamma \), and another pair of characteristics departing from \( \Gamma \) into the neighboring materials, so the disturbances may be regularly transmitted across \( \Gamma \). If, however, ineqs. (3) are violated, then the Cauchy problem with discontinuous coefficients becomes ill-posed: it demonstrates either non-existence, or non-uniqueness of a smooth solution [1, 6, 12, 13]. Physically, this is associated with the strong discontinuities in \( u, v \) themselves, i.e., the shocks that appear because of the collision of characteristics. To make the problem well-posed in this circumstances, we need additional information about the evolution of such shocks, so the whole situation becomes accordingly affected and complexified.

In our analysis, such occasions will be disregarded, and ineqs. (3) will be assumed satisfied.

This elucidation is important because, as a consequence, it eliminates many microstructures that, in other circumstances, might be admissible. As an example, a matrix made of material 1 distributed in \((z, t)\) with the circular inclusions made of material 2 (Fig. 1), is inadmissible in this context because along some portions of the circular interfaces the ineqs. (3) become violated. We conclude that these inequalities effectively narrow the class of admissible microgeometries.

Another reason that may affect the possibility of homogenization is the phenomenon of energy accumulation that may exist even in the material structures that are consistent with ineqs. (3). We illustrate this by the following example related to the dielectric DM. This example demonstrates, along with other things, a practical way to implement such formations.

Consider the following material structures assembled in space-time from two conventional isotropic dielectrics:

(1) An activated periodic laminate shown in Fig. 2a. Materials 1 and 2 with volume fractions \( m \) and \( 1 - m \) constitute a spatial pattern (with period \( d \)) moving with velocity \( V \) along the \( z \)-axis. We should stress that, in this formation, only the property pattern is brought to motion whereas materials themselves remain immovable within layers.
A doubly periodic rectangular ("checkerboard") structure with periods \( d \) and \( \tau \) in space and time (see Fig. 2b). A spatial pattern with period \( d \) of immovable materials 1 and 2 exists over the time interval \((0, n\tau)\), \(0 < n < 1\); at time \( n\tau \) the pattern exhibits a property "flip-over": where we originally had material 1, now appears material 2, and vice versa. The next "flip-over", back to the original deployment, occurs at time \( \tau \), and so on.

In a lumped parameter version, “materials 1 and 2” will be represented by the \( LC \)-pairs taking values \((L_1, C_1)\) and \((L_2, C_2)\), respectively. We construct a transmission line as a ladder of series inductors and shunt capacitors as shown in Fig. 3a, and Fig. 3b (flip over), and appropriately activate materials 1 and/or 2. Inductors are activated by using a static \( H \)-field offset on the non-linear \( B-H \) curve, and capacitors adjusted by applying varistor diodes.

By a due switching we shall be able to implement either of material formations reproduced in Figs. 2a and b. Particularly, anticipating the effect of energy accumulation, we assume that, for the checkerboard, the wave impedances \( \gamma = \sqrt{L/C} \) of both materials are the same, while the phase velocities \( a = 1/\sqrt{LC} \) are different: \( a_2 > a_1 \). As to the laminate pattern of Fig. 2a, we assume that its velocity \( V \) satisfies ineqs. (3). The structures illustrated in Figs. 2a and b are both admissible in
the sense of these inequalities, and yet they demonstrate quite a different overall performance. A moving index activated laminate shown in Fig. 2a may create conditions under which the waves much longer than a spatial period will be viewed by a laboratory observer as if they were travelling through a moving uniform dielectric. The velocity of such effective motion is different from the velocity $V$ of a moving pattern; along with other effective parameters of a laminate, it has been determined by a regular homogenization procedure described in [1, 12, 14]. Such procedure works well for laminates; actually, for a periodic laminate assembled from two distinct materials, the homogenization problem received an exact solution in [15], with the aid of the Floquet technique. It is important to note that in a laminate, the energy of long waves measured in the frame moving together with the property pattern is preserved over the material period: the energy pumped into the wave through a material switch $1 \rightarrow 2$ is released by the wave through the next switch $2 \rightarrow 1$ [12,16].

![Graph](image)

**Fig. 4** Limit cycles in the checkerboard structure.

A checkerboard formation of Fig. 2b is entirely different as a medium supporting the wave propagation. The edges of the rectangles in the checkerboard are parallel to the axes $t$ and $z$; we term those edges, respectively, as static and temporal. When an initially smooth disturbance $u(z,0)$ begins to propagate through the checkerboard, it splits into two waves that travel each along its individual characteristic within every material rectangle (the D’Alembert solution). Because the wave impedances of both materials are the same, these waves do not interact as they go from one material to another (no reflected waves on spatial interfaces and only one transmitted wave on a temporal interface). For this reason, the two waves are totally independent, and it is enough to study the performance of just one such wave. The characteristic paths are illustrated in Fig. 4 for the wave travelling from left to right; the family of paths is reproduced there for the values of parameters $a_1 = 0.6, a_2 = 1.1, m = 0.4, n = 0.5$. We see that the initial disturbance is split into a series of impulses, one impulse per material period, and every such impulse approaches its own selected characteristic (a limit cycle). After a few material periods, each impulse practically contracts to a spike. An attentive reader will notice that the characteristics in Fig. 4 always leave material 2 across the static interface, and enter it from across the temporal interface. These entrances occur twice through every material period, and at each entrance the energy of disturbances increases in the proportion $a_2/a_1 > 1$, so the energy $w_N$ after every material period becomes equal to $w_0(a_2/a_1)^2$ where $w_0$ is the energy before the wave enters this period. As to the static interfaces where the wave leaves material 2, nothing dramatic happens to the energy since the energy flux is there preserved. We see that the checkerboard offers a mechanism of a non-stop exponential growth of energy: due to a special kinematics of characteristics, the travelling wave never loses its energy, but gains it each time it enters the material 2 from across the temporal interface. This mechanism works for many extended ranges of material parameters (see [16,17]), so the set of their values chosen in Fig. 4 is by no means exceptional. Physically, the energy is pumped into the wave by an external agent; when the scale of the checkerboard shrinks to zero, the energy stored in the wave tends to infinity. Clearly, in this situation, the homogenization, in its standard version, becomes impossible. An attempt to formally solve the s.c. “cell” problem appearing in the standard homogenization scheme turns out to be unsuccessful in these circumstances. The phenomenon of “divergent energy” stays in close connection with the rectangular material geometry: to support the wave propagation through such a structure, an external agent should apply infinite energy.
With this background, the statement saying that homogenization works fine when the material assemblage becomes a spatio-temporal laminate appears to be noteworthy. The following section offers explanation for this remarkable exception.

3 Homogenization of dynamic laminates

As mentioned above, a standard asymptotic procedure [14] was implemented in [1, 12] towards activated laminates assembled from materials 1 and 2 as illustrated in Fig. 2a. The formulae for the effective properties obtained at that time were eventually confirmed in [15] as a result of an exact solution produced by the Floquet technique. The system (2) reduces, after averaging, to the form

\[ v_1 = pu_2 - qu_1, \quad v_2 = qu_2 + ru_1, \]

with the coefficients \( p, q, r \) specified as

\[
p = k_1k_2\left(\frac{T}{\rho}\right)\frac{V^2 - \frac{1}{\rho}}{V^2 - \frac{1}{\rho}},
\]

\[
q = -V\frac{k_1k_2}{\rho}\left(\frac{T}{\rho}\right) - \frac{\left(\frac{T}{\rho}\right)}{V^2 - \frac{1}{\rho}}, \tag{5}
\]

\[
r = \frac{\rho_1\rho_2}{\rho}\frac{V^2 - \bar{k}\left(\frac{T}{\rho}\right)}{V^2 - \frac{1}{\rho}}.
\]

Here the symbol \( (\cdot) \) means

\[
(\cdot) = m(\cdot) + (1 - m)(\cdot)1;
\]

the materials 1 and 2 are represented in a laminate with volume fractions \( m \) and \( 1 - m \), respectively (see Fig. 2a).

The system (4) describes propagation of the low frequency envelopes \( u, v \) of modulated waves that appear in the Floquet solution (we preserve the original symbols \( u \) and \( v \) for them). In other words, \( u, v \) appear to be the Floquet field functions averaged over a fast material period \( \epsilon \) – the period of the short wave carriers. The phase velocities associated with (4) have the meaning of the group velocities of the Floquet wave. These velocities, measured in a laboratory frame, are defined as [12]

\[
(a_{\text{eff}})_{1,2} = -V^{2}\frac{\bar{k}'\left(\frac{T}{\rho}\right) - \left(\frac{T}{\rho}\right)}{V^2 - \bar{k}\left(\frac{T}{\rho}\right)} \pm a_{12}^{2} \frac{\sqrt{(V^2 - \bar{k})\left(V^2 - \bar{k}\left(\frac{T}{\rho}\right)\right)}}{V^2 - \frac{1}{\rho}}; \tag{6}
\]

they appear to be the roots of the equation

\[
r a_{\text{eff}}^2 - 2qa_{\text{eff}} - p = 0,
\]

following from (4).

Eqs. (4)–(6) apply in both cases mentioned in ineqs. (3); the velocities (6) take finite values due to (3), and this holds true even if \( V = \infty \) when the laminates become purely temporal. Clearly, this is because the layers are activated and do not involve any background material motion. The energy of low frequency waves is not accumulated in them for reasons mentioned above in Sect. 2.

Being the low frequency limits of Floquet solutions, the envelopes \( u, v \) should be the weak limits that appear in a standard homogenization scheme. This scheme is known to work when the original fields demonstrate the weak compactness, and this property is established as a consequence of Friedrichs inequality. Such inequality follows from the boundedness of energy (or any other quadratic functional of the derivatives playing the role of energy). Fortunately, such functionals exist in the case of dynamic laminates. For the case of slow laminates \( V < a_1 \) listed first in ineqs. (3), this has been shown in [18]; for the case of fast laminates \( V > a_2 \), the quadratic functional will be demonstrated below.

(i) Case of slow (timelike) laminates

The wave equation (1) with coefficients \( \rho, k \) dependent on \((z - Vt)/\epsilon\), is transformed to the new variables

\[
\zeta = z - Vt, \quad \tau = t. \tag{7}
\]
The Lagrangian

\[ L = \frac{1}{2} \rho u^2_t - \frac{1}{2} ku^2_z - \frac{1}{2} \rho u^2_\tau - V u_\tau u_\zeta + \frac{1}{2} (\rho V^2 - k) u^2_\zeta \]  

(8)
does not explicitly depend on \( \tau \), and the laminate in the frame (7) becomes static (immovable). By a standard technique we obtain the energy equation

\[ \frac{\partial W_{\tau\tau}}{\partial \tau} + \frac{\partial W_{\tau\zeta}}{\partial \zeta} = 0, \]  

(9)

where \( W_{\tau\tau}, W_{\tau\zeta} \) are, respectively, the energy density and the energy flux density in the frame (7), calculated as

\[ W_{\tau\tau} = u_\tau \frac{\partial L}{\partial u_\tau} - L = \frac{1}{2} \rho u^2_\tau + \frac{1}{2} (k - \rho V^2) u^2_\zeta, \]  

(10)

\[ W_{\tau\zeta} = u_\tau \frac{\partial L}{\partial u_\zeta} = -\rho V u^2_\tau - (k - \rho V^2) u_\tau u_\zeta. \]  

(11)

It is easy to check that \( W_{\tau\zeta} \) is continuous across the layers’ interfaces \( \zeta = \text{const.} \); this property together with (9) yields

\[ \frac{d}{d\tau} \int_{-\infty}^{\infty} W_{\tau\tau} d\zeta = W_{\tau\zeta}\bigg|_{\zeta = -\infty}^{\zeta = \infty}; \]

the rhs vanishes if the field disappears at \( z = \pm \infty \). Therefore, the total energy

\[ E(\tau) = \int_{-\infty}^{\infty} W_{\tau\tau} d\zeta \]

measured in the frame (7) is preserved in time \( \tau : E(\tau) = E_0 = \text{const.} \). The constant \( E_0 \) is positive because (see (10))

\[ k - \rho V^2 > 0 \]

for a slow laminate. We conclude that there exists a positive constant \( \mu \) independent of \( \epsilon \) such that

\[ E_0 > \frac{1}{\mu} \int_{-\infty}^{\infty} (u^2_\tau + u^2_\zeta) d\zeta. \]

This means that there exists a subsequence \( u_{\tau n}, u_{\zeta n} \) such that \( u_{\tau n} \rightarrow u_\tau, u_{\zeta n} \rightarrow u_\zeta \), and homogenization with respect to \( \zeta \) becomes possible.

(ii) Case of fast (spacelike) laminates

To discuss this case, we introduce new variables

\[ \xi = z, \quad \theta = t - \frac{z}{V}, \]  

(12)
in the Lagrangian expressed as

\[ L = \frac{1}{2} \rho u^2_t - \frac{1}{2} ku^2_z = \frac{1}{2} \left( \rho - \frac{k}{V^2} \right) u^2_\theta + \frac{k}{V} u_\theta u_\xi - \frac{1}{2} ku^2_\xi. \]  

(13)

This expression does not depend explicitly on \( \xi \), and the lamination in coordinates (12) becomes purely temporal (dependent upon \( \theta \)). By a standard technique, we arrive at the momentum equation

\[ \frac{\partial W_{\xi\theta}}{\partial \theta} + \frac{\partial W_{\xi\xi}}{\partial \xi} = 0, \]  

(14)

with \( W_{\xi\theta}, W_{\xi\xi} \) being, respectively, the momentum density and the momentum flux density in coordinates (12). We calculate them as

\[ W_{\xi\theta} = u_\xi \frac{\partial L}{\partial u_\theta} = \frac{k}{V} u^2_\xi + \left( \rho - \frac{k}{V^2} \right) u_\theta u_\xi, \]  

(15)
\[ W_{\xi\xi} = u_{\xi} \frac{\partial L}{\partial u_{\xi}} - L = -\frac{1}{2} \left( \rho - \frac{k}{V^2} \right) u_{\theta}^2 - \frac{1}{2} ku_{\xi}^2. \] (16)

From (14) and due to the continuity of \( W_{\xi\theta} \) across the interfaces \( \theta = \text{const.} \), we obtain
\[ \frac{d}{d\xi} \int_{-\infty}^{\infty} W_{\xi\xi} d\theta = W_{\xi\theta} \bigg|_{\theta = -\infty}^{\theta = \infty}; \]

the rhs vanishes if the field disappears at \( z = \pm \infty \) for fixed \( t \).

The net momentum flux
\[ M(\xi) = \int_{-\infty}^{\infty} W_{\xi\xi} d\theta \]
in the frame (12) is therefore independent of \( \xi : M(\xi) = M_0 = \text{const.} \) This constant is negative because (see (16)) \( \rho V^2 - k > 0 \) for a fast laminate. We now conclude that there is a positive constant \( \nu \) independent of \( \epsilon \) such that
\[ M_0 < -\frac{1}{\nu} \int_{-\infty}^{\infty} \left( u_{\theta}^2 + u_{\xi}^2 \right) d\theta, \]
or, since \( M_0 < 0 \),
\[ \int_{-\infty}^{\infty} \left( u_{\theta}^2 + u_{\xi}^2 \right) d\theta < -\nu M_0, \]

This guarantees the existence of a subsequence \( u_{\theta_n}, u_{\xi_n} \), such that \( u_{\theta_n}, u_{\theta}, u_{\xi_n}, u_{\xi} \), and the homogenization with respect to \( \theta \) becomes possible. The formulae (5), (6) for the effective constants apply to both slow and fast laminates.

### 4 Conclusion

The weak compactness property sufficient for justification of the homogenization procedure hold with respect to the local fields dependent on the fast variable \( (z - Vt)/\epsilon \) in both slow and fast laminates. A laminar microstructure is in this context special because it supports, in appropriate frames, the conservation laws for the energy (momentum) expressed by Eqs. (9) and (14), respectively. For a general microstructure, however, the energy-momentum tensor does not suggest any visible conservation laws, and the homogenization procedure, in its standard version based on weak compactness, cannot, generally speaking, be justified. This statement received confirmation through the analysis of a rectangular checkerboard structure in 1D-space and time [16,17] where a disturbance travelling through such a structure was shown to demonstrate the exponential growth of energy.

### References