A dispersive effective equation for wave propagation through spatio-temporal laminates

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Received 6 September 2002; accepted 4 December 2002

Abstract

We consider the problem of wave propagation through one-dimensional spatio-temporal or dynamic laminates when the wavelength of the disturbance is large relative to the scale of the microstructure. Dynamic materials are heterogeneous formations assembled from materials which are distributed on a microscale in space and in time. Using the techniques of Floquet analysis and asymptotic expansions, we reveal the dispersive nature of the effective medium. The effects are supported by direct numerical simulation of the heterogeneous problem. These results are compared with the exact solution of the effective equation.

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1. Introduction

In this paper, we consider the problem of wave propagation through one-dimensional dynamic laminates. We derive an effective medium equation that reveals the dispersive nature of such structures.

Dynamic materials are composites in which the constituent properties, such as permeability, inductance, capacitance, density, etc., are distributed on a microscale in space and in time. Optimal material design for static or non-smart applications generally results in the formation of ordinary composites—mixtures where the design variables are position dependent but invariant in time. When it comes to dynamic applications, however, temporal variability in the material properties is needed in order to adequately match the changing environment. Therefore, we have made it our goal to study such structures in order to understand their behavior and potential applications, see [1–9].

Dynamic materials may appear in very diverse physical implementations, including mechanical and electromagnetic. Nevertheless, we may distinguish two principal ways of making them by the spatio-temporal mixing of ordinary materials via the processes of activation and kinetization [2,10]. Activated dynamic materials are obtained by instantaneous or gradual change of the material parameters, e.g. self-induction, capacitance, cross-sectional area, in various parts of the system in the absence of relative motion of those parts. Kinetic dynamic materials are obtained when various parts of the system are exposed to relative motion that is prearranged and generated in a certain way. The materials of this type can be perceived as mixtures of two or more ordinary materials that alternate in space on a microscale; this alternation occurs due to the fact that every constituent participates in its individual material...
motion. Gas bubbles in blood flow provide a very natural example of such a composite since the gas and the blood have very different material properties, and the bubbles move in the blood resulting in a composite material density, viscosity, and pressure at a particular point in space, varies in time. Generally speaking though, a dynamic material may not be purely activated or kinetic. Instead, its formation may involve both the activation and kinetization processes. Therefore, dynamic materials should be artificially constructed, and such construction requires special technological solutions. From our ongoing mathematical and numerical study of these novel materials, though, it is clear that they hold a promising means of maintaining control over the properties of the material environment that conduct the dynamical disturbances. As a result, this class of materials will prove invaluable for various engineering and industrial applications.

A dynamic disturbance on a scale much greater than the scale of a spatio-temporal microstructure will perceive a spatio-temporal formation as a new material with its own effective properties. The properties depend on the individual properties of the constituent materials, the microgeometry of the mixture, as well as the temporal arrangement. By allowing spatio-temporal variability in the material constituents, we shall be able to effectively control the dynamic processes by creating effects that are unachievable through purely spatial (static) material design.

In dynamic laminates, the material parameters are periodic, piecewise constant and move with constant velocity. It has been shown analytically and numerically in [1,4,8] that by appropriately controlling the design factors of a one-dimensional spatio-temporal laminate, the resulting d’Alembert waves may travel with different speeds. It actually becomes possible for both waves to travel in the same direction so that a section of the composite does not experience the disturbance; this is what we refer to as the screening effect. With an ordinary static composite, this effect is impossible since the waves always travel in opposite directions with the same speeds. The industrial applications of such materials whose properties one can actively change and control in order to direct and guide waves through it are vast. It has also been found that by filling a waveguide with the appropriate spatio-temporal composition of dielectric materials, we are able to eliminate the cutoff frequency for electromagnetic wave propagation. That is, disturbances of all frequencies are able to travel undamped along the waveguide; a cutoff frequency always exist in static media.

Dynamic laminates possess a periodic structure, so the response of the medium can be determined exactly using Floquet analysis for ordinary differential equations. When the period of the microstructure is small relative to the characteristic length scale of the disturbance, the Floquet results also help us to determine a homogenized description of the material. In [4], this process is used to determine a second order partial differential equation that approximates the hyperbolic nature of the medium. In [11], the authors obtain an effective medium description of the static, periodic medium which exhibits dispersion for large times. In the current article, we establish the dispersive effective medium model for spatio-temporal laminates. We find that dispersive effects become relevant even earlier for the dynamic problem.

The outline of this paper is as follows. We begin the following section by describing our model problem. We then use Floquet analysis and an asymptotic expansion to derive a fourth order effective medium equation that governs the behavior of wave motion through the spatio-temporal laminate. In Section 3, we outline the numerical methods that are used to compute numerical solutions of the heterogeneous problem. A presentation and discussion of the results from such simulations is given in Section 4, and the paper concludes with a summary of our findings.

2. Derivation of effective medium equation for dynamic laminates

2.1. Problem statement

We model wave motion through dynamic laminates by the second order linear wave equation

\[ (\mu \psi_t)_t - (\varepsilon \psi_x)_x = 0 \]  

(1)
with initial conditions

\[ z(x, 0) = f(x), \quad z_t(x, 0) = g(x). \]

The coefficients \( \rho \) and \( k \) are fast periodic functions given by

\[ \rho(x, t) = \rho \left( \frac{x - V t}{\varepsilon} \right), \quad k(x, t) = k \left( \frac{x - V t}{\varepsilon} \right), \]

where \( \varepsilon \) is a small parameter. We are interested in the case where \( \varepsilon \) is small relative to the wavelength of the disturbance. The property pattern moves with uniform constant velocity \( V \) and \( \rho(\cdot) \) and \( k(\cdot) \) are 1-periodic functions. At \( \varepsilon = 0 \), the density and stiffness have a periodic pattern, and (2) may be perceived as this pattern being activated, i.e., brought into motion with velocity \( V \) along the \( x \)-axis. These assumptions may be listed as follows:

(a) At each point \((x, t)\), the controls \( \rho \) and \( k \) can take either the values \((\rho_1, k_1)\) or \((\rho_2, k_2)\); we refer to these as ‘material 1’ and ‘material 2’.

(b) These materials are placed within alternating layers having the slope \( d \) on the \((x, t)\)-plane.

(c) The period of the pattern is composed of two successive layers filled, respectively, by materials 1 and 2, the volume fractions of these layers being \( \varepsilon_1 \) and \( \varepsilon_2 \).

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The pattern velocity \( V \) is chosen so as to ensure the regular transition of continuous disturbance \((x, t)\) through the interface from one layer to another. For smooth solutions, kinematic and dynamic compatibility conditions of continuity across the material interfaces are enforced. These conditions are satisfied if the following relationship holds:

\[ V^2 < a_1^2 \quad \text{or} \quad V^2 > a_2^2. \]

We refer to these two regimes as the subsonic or slow range regime, and the supersonic or fast regime where \( |V| > a_1, a_2 \). Static laminates \((V = 0)\) fall into the subsonic category; this problem has been quite well studied. The temporal laminate problem \((V = \infty)\), studied in [9] is the limiting laminate structure of the supersonic class.

In [1], Lurie uses a standard analytical homogenization procedure to calculate the effective phase velocities, and in [4], he derives the effective second order differential equation for a dynamic laminate when the period of the medium is much smaller than the wavelength of the initial disturbance. Using the notation \( \langle \zeta \rangle = m_1 \zeta_1 + m_2 \zeta_2 \) and \( \xi = m_1 \zeta_1 + m_2 \zeta_2 \), we have the following differential equation for \( \zeta \), the value of the disturbance \( z \) averaged over the period of the array:

\[ \frac{1}{\sigma_{\zeta}} \left[ V^2 - k \left( \frac{1}{\rho} \right) \right] \langle \zeta \rangle_{x} + 2V \left( \frac{1}{k} \right) [\rho \phi - a^2] \langle \xi \rangle_{x} - \rho \left( \frac{1}{k} \right) \left[ V^2 - \frac{1}{k} \left( \frac{1}{\rho} \right) \right] \langle \zeta \rangle_{xx} = 0. \]

When \( V \) satisfies (3), the equation above is hyperbolic. Like its uniform material counterpart, this dynamic material will allow for d’Alembert wave solutions travelling with characteristic phase velocities.

The second order scalar equation (1) can be written as a first order system by introducing the auxiliary variable, \( v \)

\[ \rho v_x = v_x, \]

\[ v_t = k z_x. \]

The compatibility conditions are then continuity of \( z \) and \( v \) across the material interfaces. The characteristic form of these equations is

\[ \begin{pmatrix} z + \frac{v}{\mu} & 0 & z + \frac{v}{\mu} \\ -a & 0 & a \\ 0 & a & 0 \end{pmatrix} \begin{pmatrix} z + \frac{v}{\mu} \\ 0 \\ 0 \end{pmatrix} = 0. \]

[84x525] (b) These materials are placed within alternating layers having the slope \( d \) on the \((x, t)\)-plane.

[84x385] The temporal laminate problem \((V \to \infty)\) is studied in [9] is the limiting laminate structure of the supersonic class.

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\[ \begin{pmatrix} z + \frac{v}{\mu} & 0 & z + \frac{v}{\mu} \\ -a & 0 & a \\ 0 & a & 0 \end{pmatrix} \begin{pmatrix} z + \frac{v}{\mu} \\ 0 \\ 0 \end{pmatrix} = 0. \]
where $\mu = \sqrt{\rho k}$ is called the ‘material impedance’. Information $z - v/\mu_i$ is advected with velocity $-a_i$ through material $i$, and $z - v/\mu_i$ is advected with velocity $a_i$ through material $i$.

In the appendix of [8], it is shown that a wave moving from material $\alpha$ to material $\beta$ across a material interface that is itself moving with speed $|V| < a_\alpha, a_\beta$ splits into two waves. There is the reflected wave which moves back into material $\alpha$, and the transmitted wave travelling into material $\beta$. The amplitudes of the resulting waveforms relative to that of the incident one are given by the reflection and transmission coefficients $T_{\alpha,\beta}, R_{\alpha,\beta}$:

$$T_{\alpha,\beta} = \frac{2\mu_\alpha}{\mu_\alpha + \mu_\beta}, \quad R_{\alpha,\beta} = \frac{\mu_\alpha - \mu_\beta}{\mu_\alpha + \mu_\beta}.$$  

For the supersonic problem, however, we have calculated that an incident wave splits into two transmitted waves with transmission coefficients

$$T_{\alpha,\beta} = \frac{\mu_\beta + \mu_\alpha}{2\mu_\beta}, \quad \tilde{T}_{\alpha,\beta} = \frac{\mu_\beta - \mu_\alpha}{2\mu_\beta},$$

see Fig. 1.

Note that when the material impedances are the same, the incident wave does not split as it passes through the interface and has no change in amplitude. Therefore, we should not expect to see any dispersion effects in this case.

It can also be shown that when $0 < V < \infty$ and $a_\alpha \neq a_\beta$, the frequencies of the newborn waves differ from that of the incident wave.

The repeated reflection and scattering of waves through heterogeneous materials results in local dispersion of the information travelling through. These small scale dispersions evidently manifest themselves on a larger scale in finite time. Since the materials under consideration have a clear, periodic structure, we can use the techniques of Floquet analysis to capture and describe the dispersive nature of the effective wave motion.

2.2. Derivation of the higher order effective equation

In this section, we derive a fourth order effective equation for one-dimensional dynamic laminates.

We make a change of variables $\xi = x - Vt$, $\tau = t$ to convert the governing system of equations (5) and (6) to

$$z_{\tau} = Vz_{\xi} + \frac{1}{\rho}q_{\xi}, \quad \tau = V\xi + \xi\tau_\xi.$$  

Now $\rho$ and $\xi$ depend on $\xi$ only and are $\tau$-periodic. Performing the Fourier transform in $\tau$ using $p$ as the Fourier variable

$$z(\xi, \tau) = \int e^{ip\tau} \tilde{z}(\xi, p) \, dp,$$  

Fig. 1. Wave splitting at: (a) a subsonic interface; (b) a supersonic interface.
one obtains
\[ V\xi + \frac{1}{p}\xi = ip\xi, \]
\[ V\xi + k\xi = ip\xi. \]
Over one period, the solution is
\[ \bar{z}(t, p) = \hat{A} e^{\nu_\xi^+ t} + \hat{B} e^{\nu_\xi^- t}, \]
\[ \hat{v}(t, p) = -\nu_\xi \hat{A} e^{\nu_\xi^+ t} + \nu_\xi \hat{B} e^{\nu_\xi^- t}, \]
where \( m_2 \xi \leq \xi \leq 0, \)
and
\[ \hat{z}(t, p) = \hat{C} e^{\nu_\xi^+ t} + \hat{D} e^{\nu_\xi^- t}, \]
\[ \hat{v}(t, p) = -\nu_\xi \hat{C} e^{\nu_\xi^+ t} + \nu_\xi \hat{D} e^{\nu_\xi^- t}, \]
where \( u_i = ip/(V - a_i), \beta_i = ip/(V + a_i), \) and \( \mu_i = \sqrt{\alpha_i} \) is the material impedance.

By Floquet’s theorem for differential equations with periodic coefficients [12], the general solution \( \hat{z} \) of (8) and (9) is
\[ \hat{z}(t, p) = c_1 e^{\nu_\xi^+ t} + c_2 e^{\nu_\xi^- t} + c_3 e^{\nu_\xi^+ t - \beta_1 t} + c_4 e^{\nu_\xi^- t - \beta_2 t}, \]
\[ \hat{v}(t, p) = c_1 e^{\nu_\xi^+ t} - c_2 e^{\nu_\xi^- t} + c_3 e^{\nu_\xi^+ t - \beta_1 t} - c_4 e^{\nu_\xi^- t - \beta_2 t}, \]
where \( v_\xi = v_\xi(p) \) are the Floquet exponents, \( f_{\gamma}, g_{\gamma} \) are \( e^\gamma \)-periodic functions of \( \xi \), and \( c_{\gamma} \) are arbitrary constants.

Eq. (10) expresses the solution as short wave carriers \( (f_{\gamma}) \) being modulated by long wave factors \( e^{\nu_\xi \gamma} \). The process of averaging the solution over the period of the microstructure thus detects the low frequency envelopes and eliminates the high frequency carriers. To characterize effective wave motion, it is therefore necessary to obtain expressions for the Floquet exponents.

The interface conditions of continuity of \( z \) and \( v \) are
\[ \hat{z}(0^+, p) = \hat{z}(0^-, p), \]
\[ \hat{v}(0^+, p) = \hat{v}(0^-, p), \]
\[ \hat{z}(m_1^+, p) = \hat{z}(m_1^-, p), \]
\[ \hat{v}(m_1^+, p) = \hat{v}(m_1^-, p), \]
etc. From (10) and (11), \( \hat{z}(\xi, p) = e^{\nu_\xi^+ \gamma (\xi - \eta, p)} \), and similarly for \( \hat{v}(\xi, p) \), so the coefficients \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) must be solutions of the system
\[ \hat{A} + \hat{B} = \hat{C} + \hat{D}, \]
\[ -\mu_2 \hat{A} + \nu_\xi \hat{B} = -\mu_1 \hat{C} + \mu_1 \hat{D}, \]
\[ e^{\nu_\xi^+} \hat{A} e^{-\nu_\xi^+ (\xi - \eta)} + e^{\nu_\xi^-} \hat{B} e^{-\nu_\xi^- (\xi - \eta)} = e^{\nu_\xi^+ (\xi - \eta)} \hat{C} + e^{\nu_\xi^- (\xi - \eta)} \hat{D}, \]
\[ -\mu_2 e^{\nu_\xi^+} \hat{A} e^{-\nu_\xi^+ (\xi - \eta)} + \mu_2 e^{\nu_\xi^-} \hat{B} e^{-\nu_\xi^- (\xi - \eta)} = -\mu_1 e^{\nu_\xi^+ (\xi - \eta)} + \mu_1 e^{\nu_\xi^- (\xi - \eta)} \hat{D}. \]
This system has a solution if and only if the determinant of the system is zero. The Floquet exponents \( v = v_\xi(p) \) will therefore solve
\[ Y^2 - 2\gamma Y + \gamma^2 = 0, \]
where \( Y = e^{\nu_\xi^+} \),
\[ H = -\frac{1}{2}R \sin(\theta_1) \sin(\theta_2) + \cos(\theta_1) \cos(\theta_2), \]
\[ \theta_i = p \sqrt{m_i}, \]
\[ R = \frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1}, \]
and
\[ \gamma = e^{\nu_\xi^+ (\theta_1 + \theta_2)}. \]
Let \( \chi \) be defined by
\[
\cosh(\chi) = H = -\frac{1}{2} R \sin(\theta_1) \sin(\theta_2) + \cos(\theta_1) \cos(\theta_2),
\] (14)
then, \( Y = \chi H \leq \gamma (\gamma^2 - 1)^{1/2} \), so
\[
Y = e^{\chi} = \gamma (1 + \sinh \chi) = \gamma e^{\frac{\chi}{2}}.
\]
From (12),
\[
v_2 \varepsilon = i V \left( \frac{\theta_1}{a_1} + \frac{\theta_2}{a_2} \right) \pm \chi,
\] (15)
for \( \theta_1 \) small, \( H \) is just under 1 thus \( \chi \) and \( \varepsilon \) are imaginary.
Using the first few terms of the Maclaurin expansions of the trigonometric terms in (14) gives
\[
\cosh(\chi) \approx 1 + E_2(\varepsilon p)\theta_1^2 + E_4(\varepsilon p)\theta_1^4.
\] (16)
for \( \theta_1 \) small, where
\[E_2 = -\frac{1}{4} [ \tilde{\theta}_1^2 \bar{\theta}_2 + \bar{\theta}_2^2 + \tilde{\theta}_2^2],
\] (17)
\[E_4 = \frac{1}{4} [ \tilde{\theta}_1^4 + \bar{\theta}_1^2 \bar{\theta}_2^2 + 2 \tilde{\theta}_1^2 \bar{\theta}_2^2],
\] (18)
and \( \tilde{\theta}_1 = \theta_1 / \varepsilon \). Note that \( E_2 < 0 \) and \( E_4 > 0 \). Expanding the left-hand side of (16) gives
\[
1 + \frac{\theta_1^2}{2} + \frac{\theta_1^4}{24} \approx 1 + E_2(\varepsilon p)\theta_1^2 + E_4(\varepsilon p)\theta_1^4 \implies \chi^2 + 12\chi^4 + 24E_2(\varepsilon p)\theta_1^2 + 24E_4(\varepsilon p)\theta_1^4
\]
\[
\implies \chi^2 = 2E_2(\varepsilon p)\theta_1^2 + \left[ 2E_4 - \frac{E_2^2}{3} \right] (\varepsilon p)^2 \implies \chi^2 = 2E_2(\varepsilon p)\theta_1^2 + \left[ 2E_4 - \frac{E_2^2}{3} \right] (\varepsilon p)^2. \] (19)

Writing \( \lambda \varepsilon = \varepsilon \) and using (15), we get
\[
\lambda \varepsilon = i V \left( \frac{\theta_1}{a_1} + \frac{\theta_2}{a_2} \right) \pm \chi \implies i \lambda \varepsilon - V \left( \frac{\theta_1}{a_1} + \frac{\theta_2}{a_2} \right) (p) = \pm \chi
\]
\[
\implies -\lambda^2 \varepsilon^2 + 2 \lambda V \left( \frac{\theta_1}{a_1} + \frac{\theta_2}{a_2} \right) p \varepsilon^2 - V^2 \left( \frac{\theta_1}{a_1} + \frac{\theta_2}{a_2} \right)^2 (p) = \chi^2,
\]
hence
\[
[2E_4 - \frac{E_2^2}{3}](p)^2 + [2E_2 + Q^2](p)^2 - 2\varepsilon Q(p) + \lambda^2 \varepsilon^2 = 0,
\] (20)
from (19) where \( Q = V(\tilde{\theta}_1 / a_1 + \bar{\theta}_2 / a_2) \). Write this as \( Ap^2 + Bp + C \) and \( \lambda^2 = 0 \), where
\[
A = 2E_4 - \frac{E_2^2}{3}, \quad B = 2E_2 + Q^2, \quad C = -2\varepsilon Q.
\]
In order to obtain an explicit expression for \( p \) in terms of \( \lambda \), we assume that the dependence of the Fourier variable \( p \) on \( \varepsilon \) is of the power series form
\[
p = c_0 + c_1 \varepsilon + c_2 \varepsilon^2 + c_3 \varepsilon^3 + \cdots.
\]
Therefore,
\[
\begin{align*}
p^2 &= c_0^2 + (2c_0c_1)\varepsilon + (2c_0c_2 + c_1^2)\varepsilon^2 + (2c_0c_3 + 2c_1c_2)\varepsilon^3 + \cdots, \\
p^4 &= c_0^4 + (4c_0c_1c_2 + 4c_1^2c_2 + 6c_0^2c_3 + 4c_1c_3 + 4c_0c_2^2)\varepsilon^2 + \cdots.
\end{align*}
\]
Inserting the ansatz into (20) and equating coefficients of powers of \( \varepsilon \), we get

\[
\begin{align*}
\text{O}(1) & : \quad Bc_0^2 + Cc_0 + \lambda^2 = 0, \\
\text{O}(\varepsilon) & : \quad B(2c_0\varepsilon) + Cc_1 = 0, \\
\text{O}(\varepsilon^2) & : \quad A_0 c_0 + B(2c_0\varepsilon^2 + c_1^2) + Cc_2 = 0, \\
\text{O}(\varepsilon^3) & : \quad A(c_0\varepsilon^3) + B(2c_0\varepsilon^3 + 2c_1\varepsilon^2) + Cc_3 = 0.
\end{align*}
\]

From this, \( c_1 = c_2 = 0 \), and

\[
\begin{align*}
c_0 &= -\frac{C \pm \sqrt{C^2 - 4BC}}{2B}, \\
c_0 &= -\frac{A\rho_0^2}{\rho_1^2}, \\
c_0 &= -\frac{2\sqrt{2}}{A\rho_0}, \quad \lambda \equiv \pm \Gamma_0 \lambda^1.
\end{align*}
\]

We now have a relationship between each of the two possible Floquet exponents \( \lambda = \pm \Gamma_0 \lambda^1 \) and the Fourier variable \( p = \Omega_0 \rho_0 \pm \Gamma_0 \lambda^1 \varepsilon^2 + O(\varepsilon^4) \).

We will obtain expressions for \( A, \Omega, Q^2 + 2E_2, \) and \( \sqrt{2E_2} \). First, we recognize two identities that we will use later:

\[
\begin{align*}
R m_1 (m_2 + m_1) + m_1^2 a_1^2 + m_2^2 a_2^2 &= m_1 m_2 \left( \frac{k_1}{\rho_2} + \frac{k_2}{\rho_1} \right) + m_1^2 \frac{k_1}{\rho_1} + m_2^2 \frac{k_2}{\rho_2}, \\
&= \left( \frac{m_1}{\rho_2} + \frac{m_2}{\rho_1} \right) (m_1 k_1 + m_2 k_2) + \frac{m_1}{\rho_2} + \frac{m_2}{\rho_1},
\end{align*}
\]

(21)

Remember the notation \( \tilde{p} = m_1\beta_1 + m_2\beta_2 \) and \( \tilde{p} = m_1\beta_1 + m_2\beta_2 \). From Eq. (17),

\[
-2E_2 = \tilde{p} k_1 b_1 + \tilde{p} b_2 = \frac{N}{(V^4 - a_1^2)(V^4 - a_2^2)}.
\]

where

\[
N = R m_1 (m_2 + m_1) (V^4 - a_1^2) (V^4 - a_2^2) + m_1^2 a_1^2 (V^4 - a_1^2) + m_2^2 a_2^2 (V^4 - a_1^2) + m_1^2 a_1^2 (V^4 - a_2^2) + m_2^2 a_2^2 (V^4 - a_2^2)
\]

The coefficient of \( V^4 \) in \( N \) is given by (21), the constant term is (22), and the coefficient of \( V^2 \) in \( N \) is

\[
-\tilde{a}_1^2 \left[ R m_1 m_2 \left( \frac{a_1}{\rho_2} + \frac{a_2}{\rho_1} \right) + 2m_1^2 + 2m_2^2 \right]
\]

(22)
Hence,

\[-2E_2 = \frac{a_4^2 (\bar{\theta}V - \tilde{\theta})}{(V^2 - a_4^2)^2} \cdot \frac{(\bar{\theta}V - \tilde{\theta})(\bar{\theta}V + \tilde{\theta})}{(V^2 - a_4^2)^2} \]

and

\[\sqrt{-2E_2} = \frac{a_4^2 \theta}{(V^2 - a_4^2)(V^2 - a_4^2)^2} \sqrt{(\bar{\theta}V - \tilde{\theta}) \left( \left( \frac{k}{\rho} \right)^2 V^2 - \left( \frac{k}{\rho} \right)^2 \right)}.

From Eqs. (17) and (18),

\[A = 2E_4 \cdot \frac{\mu_2^2}{\mu_1^2} = \frac{1}{12} \left[ \tilde{\theta}_1^2 + \tilde{\theta}_2^2 + 6\tilde{\theta}_1^2 \tilde{\theta}_2^2 + 2R\tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 \right] - \frac{1}{12} \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

\[= \frac{1}{12} \left[ \tilde{\theta}_1^2 + \tilde{\theta}_2^2 + 6\tilde{\theta}_1^2 \tilde{\theta}_2^2 + 2R\tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 \right] - \frac{1}{12} \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

\[= \frac{1}{12} \frac{1}{(V^2 - a_4^2)^2} \left[ \frac{1}{12} \left( \tilde{\theta}_1^2 + \tilde{\theta}_2^2 + 6\tilde{\theta}_1^2 \tilde{\theta}_2^2 + 2R\tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 \right) - \frac{1}{12} \left( \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right) \right] \]

\[= \frac{1}{12} \left[ (V^2 - a_4^2)^2 (V^2 - a_4^2)^2 \right] \left[ \tilde{\theta}_1^2 + \tilde{\theta}_2^2 + 6\tilde{\theta}_1^2 \tilde{\theta}_2^2 + 2R\tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 \right] - \frac{1}{12} \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

\[= \frac{1}{12} \left[ (V^2 - a_4^2)^2 (V^2 - a_4^2)^2 \right] \left[ \tilde{\theta}_1^2 + \tilde{\theta}_2^2 + 6\tilde{\theta}_1^2 \tilde{\theta}_2^2 + 2R\tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 \right] - \frac{1}{12} \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

and

\[Q^2 + 2E_2 = V^2 \left( \frac{\mu_1 \mu_2}{\mu_1^2} + \frac{\mu_2 \mu_1}{\mu_2^2} \right)^2 - \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

\[= V^2 \left[ \frac{\mu_1 \mu_2}{\mu_1^2} + \frac{\mu_2 \mu_1}{\mu_2^2} \right] - \left[ \tilde{\theta}_1 \tilde{\theta}_2 + \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right] \]

\[= \frac{m_1^2}{(V^2 - a_4^2)} + \frac{m_2^2}{(V^2 - a_4^2)} + \frac{2m_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{m_1^2}{(V^2 - a_4^2)} + \frac{m_2^2}{(V^2 - a_4^2)} + \frac{2m_1m_2}{(V^2 - a_4^2)} - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{m_1^2}{(V^2 - a_4^2)} + \frac{m_2^2}{(V^2 - a_4^2)} + \frac{2m_1m_2}{(V^2 - a_4^2)} - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{m_1^2}{(V^2 - a_4^2)} + \frac{m_2^2}{(V^2 - a_4^2)} + \frac{2m_1m_2}{(V^2 - a_4^2)} - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{1}{(V^2 - a_4^2)} \left[ m_1^2 + m_2^2 - a_4^2 \right] - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{1}{(V^2 - a_4^2)} \left[ m_1^2 + m_2^2 - a_4^2 \right] - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{1}{(V^2 - a_4^2)} \left[ m_1^2 + m_2^2 - a_4^2 \right] - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{1}{(V^2 - a_4^2)} \left[ m_1^2 + m_2^2 - a_4^2 \right] - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

\[= \frac{1}{(V^2 - a_4^2)} \left[ m_1^2 + m_2^2 - a_4^2 \right] - \frac{Rm_1m_2}{(V^2 - a_4^2)} \]

(24)
Here have larger velocities than \( \langle \text{solution} \rangle \). Also, the laminate problem \( \Gamma \) Note that \( \text{and} \) materials \( (V) \). Since \( \bar{\Sigma} \) and the fourth order effective equation for wave motion through a one-dimensional spatio-temporal laminate is

\[
\begin{align*}
Q &= \frac{m_1}{V^2 - \bar{a}_1^2} + \frac{m_2}{V^2 - \bar{a}_2^2} = V \left[ \frac{m_1(V^2 - \bar{a}_1^2) + m_2(V^2 - \bar{a}_2^2)}{(V^2 - \bar{a}_1^2)(V^2 - \bar{a}_2^2)} \right] = V \frac{V^2 - (m_1V^2 + m_2V^2)}{(V^2 - \bar{a}_1^2)(V^2 - \bar{a}_2^2)},
\end{align*}
\]

From (23)–(25), we find

\[
\Omega_z = Q \pm \sqrt{\frac{-2\bar{\Sigma} \pm \Omega^2}{2\bar{\Sigma} + Q^2}} = \frac{V(V^2 - \bar{a}_2^2) \pm \bar{a}_1 \bar{a}_2 \left( \sqrt{\bar{a}_1^2 - \bar{a}_2^2} - \bar{a}_1 \right)}{V^2 - \left( \bar{a}_1^2 - \bar{a}_2^2 \right)},
\]

and

\[
\Gamma_\scalebox{0.5}{2} = \frac{\Omega^2_1}{2} - \frac{A}{\sqrt{-2\bar{\Sigma}}} = \frac{\Omega_1^2 \left( \left( \mu_1 \mu_2 - \mu_2 \mu_1 \right) |m_1|^2 \bar{a}_1 \bar{a}_2 \right)}{24(V^2 - \bar{a}_1^2)(V^2 - \bar{a}_2^2) \left( \sqrt{\bar{a}_1^2 - \bar{a}_2^2} - \bar{a}_1 \right)}.
\]

Note that \( \Gamma_\scalebox{0.5}{2} \) is zero when there is no contrast in material impedance, i.e. when \( \mu_1 = \mu_2 \). Given

\[
p = \Omega, \lambda + \Gamma_\scalebox{0.5}{2} \bar{a}_1 \bar{a}_2, \quad p = \Omega, \lambda - \Gamma_\scalebox{0.5}{2} \bar{a}_1 \bar{a}_2,
\]

we use the Floquet solution in (10) to reconstruct the expression for the fundamental wave solutions of the dynamic laminate problem

\[
z(x, t) = z(x, \tau) = \int \hat{z}(x, \rho) e^{i\rho t} d\rho \int \exp \left( i\int \left( 242 \rho \left( \frac{\bar{a}_1 \bar{a}_2 \sqrt{\bar{a}_1^2 - \bar{a}_2^2}}{V^2 - \bar{a}_1^2 - \bar{a}_2^2} - \bar{a}_1 \right) \right) t \right) f_\scalebox{0.5}{2}(x - Vt, \rho) d\rho
\]

Here \( \Sigma_\scalebox{0.5}{2} = -\Omega_2 + V \).

\[
\Sigma_\scalebox{0.5}{2} = V(242 \rho \left( \frac{\bar{a}_1 \bar{a}_2 \sqrt{\bar{a}_1^2 - \bar{a}_2^2}}{V^2 - \bar{a}_1^2 - \bar{a}_2^2} - \bar{a}_1 \right) t) f_\scalebox{0.5}{2}(x - Vt, \rho) d\rho.
\]

Since \( \hat{f} \) is \( \rho \)-periodic and represents the high frequency filling of the homogenized wave packets, the homogenized solution \( \langle z \rangle \) which is \( z \) averaged over period \( \rho \) is controlled by the exponential term in (29). Consequently,

\[
\langle z \rangle_x + \Sigma_\scalebox{0.5}{2} \langle z \rangle_x = -\Gamma_\scalebox{0.5}{2} \langle z \rangle_\scalebox{0.5}{2}, \quad \langle z \rangle_x + \Sigma_\scalebox{0.5}{2} \langle z \rangle_x = \Gamma_\scalebox{0.5}{2} \langle z \rangle_\scalebox{0.5}{2},
\]

and the fourth order effective equation for wave motion through a one-dimensional spatio-temporal laminate is

\[
\frac{\partial^2 \langle z \rangle_\scalebox{0.5}{2}}{\partial t^2} + \left( \Sigma_\scalebox{0.5}{2} + \Sigma_\scalebox{0.5}{2} \right) \frac{\partial^2 \langle z \rangle_\scalebox{0.5}{2}}{\partial x^2} + \left( \Sigma_\scalebox{0.5}{2} + \Sigma_\scalebox{0.5}{2} \right) \frac{\partial^2 \langle z \rangle_\scalebox{0.5}{2}}{\partial x^2} + \left( \Gamma_\scalebox{0.5}{2} - \Gamma_\scalebox{0.5}{2} \right) \frac{\partial^2 \langle z \rangle_\scalebox{0.5}{2}}{\partial x^2} + \left( \Gamma_\scalebox{0.5}{2} - \Gamma_\scalebox{0.5}{2} \right) \frac{\partial^2 \langle z \rangle_\scalebox{0.5}{2}}{\partial x^2} = 0.
\]

The d’Alembert wave velocities are \( \Sigma_\scalebox{0.5}{2} \) and \( \Sigma_\scalebox{0.5}{2} \). The dispersed ‘+waves’ have smaller velocities than \( \Sigma_\scalebox{0.5}{2} \) (velocities are \( \Sigma_\scalebox{0.5}{2} - \Gamma_\scalebox{0.5}{2} \scalebox{0.5}{2} \), thus the dispersion train should appear to the left of the main wave; the dispersed ‘−waves’ have larger velocities than \( \Sigma_\scalebox{0.5}{2} \), thus the dispersion train should appear to the right of the main wave.

This result should be compared with the second order equation in (4), and the higher order equation for static materials \( (V = 0) \) in [11].
2.3. Features of note

The product of the d’Alembert wave velocities is

$$\Sigma_+ \Sigma_- = -\frac{1}{\kappa} \frac{j a_i^2 V^2 - (1/j)(\kappa/\rho)}{V^2 - k(1/\rho)}.$$  \hspace{1cm} (34)

Assume that \(a_1 \leq a_2\). In [1], it is shown that

$$\frac{1}{\rho(\kappa/\rho)} < \frac{\hat{\kappa}}{k}, \quad a_1^2 < \frac{\hat{\kappa}}{k},$$

so, if the parameters of the laminate \((\rho, k, m_i)\) are such that \(1/\rho(\kappa/\rho) < a_1^2\) or \(a_1^2 < \rho(\kappa/\rho)\), then when

$$\frac{1}{\rho(\kappa/\rho)} < V^2 < a_1^2 \quad \text{or} \quad a_1^2 < V^2 < \frac{\hat{\kappa}}{k},$$

the product in (34) is positive. In such a case, the d’Alembert waves travel in the same direction—‘coordinated wave motion’. This gives rise to what we call the ‘screening effect’. For static laminates, \(V = 0\), the waves travel always in opposite directions with the same speed \((\Sigma_+ = -\Sigma_-)\).

We next look at some important limits for the effective values, \(\Omega_\pm, \Gamma_\pm, \Sigma_\pm\) as given in (26), (27) and (30). First, it is straightforward to calculate that

\[
\begin{align*}
\omega_1 \omega_2 & \left[ \frac{\omega_1^2 - \hat{\kappa}}{\omega_1^2} \left( \frac{1}{\rho} \omega_1^2 - \frac{1}{\rho} \right) \right] = m_1 a_1 (a_1^2 - a_2^2), \\
\omega_1 \omega_2 & \left[ \frac{\omega_2^2 - \hat{\kappa}}{\omega_2^2} \left( \frac{1}{\rho} \omega_2^2 - \frac{1}{\rho} \right) \right] = m_2 a_2 (a_1^2 - a_2^2),
\end{align*}
\]

and

\[
\omega_1 (a_1^2 - \hat{\kappa}) = m_1 a_1 (a_1^2 - a_2^2), \quad \omega_2 (a_2^2 - \hat{\kappa}) = m_2 a_2 (a_1^2 - a_2^2).
\]

So, from (26), when \(V = \min(a_1, a_2)\) then \(\Omega_+\) is zero and \(\Omega_-\) finite non-zero; similarly, when \(V = \max(a_1, a_2)\) then \(\Omega_-\) is zero and \(\Omega_+\) finite non-zero. Furthermore, using this in (27) and (30), one has

\[
\begin{align*}
\Sigma_+ & \to V, \quad \Gamma_+ \to 0, \quad \Gamma_- \text{ diverges as } V \to \min(a_1, a_2)^-; \\
\Sigma_- & \to V, \quad \Gamma_+ \to 0, \quad \Gamma_- \text{ diverges as } V \to \max(a_1, a_2)^+.
\end{align*}
\hspace{1cm} (35)

when \(a_1 \neq a_2\), and

\[
\begin{align*}
\Sigma_+ & \to V, \quad \Gamma_+ \to 0 \text{ as } V \to a_1, \\
\Sigma_- & \to V, \quad \Gamma_\pm \to 0 \text{ as } V \to a_1.
\end{align*}
\hspace{1cm} (36)

when \(a_1 = a_2\). Secondly, when \(\hat{\kappa}(1/\rho) > \max(a_1^2, a_2^2)\),

\[
\sqrt{\frac{\hat{\kappa}}{k} \left( \frac{1}{\rho} \right) \left( \frac{1}{\rho} - \hat{\kappa} \right)} = \omega_1 \omega_2 \sqrt{\frac{\hat{\kappa}}{k} \left( \frac{1}{\rho} \right) \left( \frac{1}{\rho} - \hat{\kappa} \right)},
\]

so from (26), \(\Omega_+\) diverges as \(V^2 \to \hat{\kappa}(1/\rho)\), but \(\Omega_-\) is finite. Therefore, by (27),

\[
\Gamma_\pm \text{ diverges as } V^2 \to \frac{\hat{\kappa}}{k} \left( \frac{1}{\rho} \right). \hspace{1cm} (37)
\]
3. Numerical method

We use an upwind numerical scheme to perform direct numerical computations of the solution to the heterogeneous, hyperbolic laminate problem described at the beginning of Section 2. The computational grid in \( x-t \) space is arranged so that material properties are constant within a space–time grid cell and change across the cell interfaces only. The grid points lie in the centers of grid cells of width \( \Delta x_i \) and \( \Delta t \) such that a layer of material \( i \) has a given number of cells of width \( \Delta x_i \) for \( i = 1, 2 \). The grid center at time \( t_n = n\Delta t \) of cell \( j \) is denoted by \( x_j^n \). The grid points move with the pattern velocity, \( V \) such that \( x_j^n = x_j^0 + V n \Delta t \), is the location of the \( j \)th grid point at time \( t_n \). The values of \( z \) and \( v \) at the grid points, denoted by \( z_j^n, v_j^n \), simultaneously represent the approximations to \( z \) and \( v \) at those points, as well as the values of \( z \) and \( v \) averaged over cell \( j \). We can equivalently think of this arrangement as that of a static laminate in \( \xi-\tau \) coordinates with wave motion governed by the system

\[
z_t = V\xi + \frac{1}{\rho}v_t,
\]

\[
v_t = V\xi + \xi \xi.
\]

Then, \( x_j^n = \xi_j^n \) for all \( n \).

The upwind method comes from integrating the system (38) and (39) over the \( j \)th \( \xi-\tau \) grid cell:

\[
z_j^{n+1} = z_j^n + \frac{1}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [V(\xi_{j+1/2}, t) - v(\xi_{j-1/2}, t)] d\tau + \frac{1}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [z(\xi_{j+1/2}, t) - z(\xi_{j-1/2}, t)] d\tau,
\]

\[
v_j^{n+1} = v_j^n + \frac{1}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [v(\xi_{j+1/2}, t) - v(\xi_{j-1/2}, t)] d\tau + \frac{1}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [v(\xi_{j+1/2}, t) - v(\xi_{j-1/2}, t)] d\tau.
\]

The scheme is

\[
z_j^{n+1} = z_j^n + \frac{\Delta t}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [V(\xi_{j+1/2} - \xi_{j-1/2})] + \frac{\Delta t}{\Delta x_j} [V(\xi_{j+1} - \xi_{j-1}) - z(\xi_{j-1}, t)],
\]

\[
v_j^{n+1} = v_j^n + \frac{\Delta t}{\Delta x_j} \int_{\tau}^{\tau+\Delta t} [V(\xi_{j+1/2} - \xi_{j-1/2})] + \frac{\Delta t}{\Delta x_j} [V(\xi_{j+1} - \xi_{j-1}) - v(\xi_{j-1}, t)].
\]

The average values of \( z \) and \( v \) along the cell interfaces \( \xi_{j+1/2}, \xi_{j-1/2} \) are calculated in a straightforward manner by tracing characteristics. The characteristic form of Eqs. (5) and (6) has been given in (7) which can be expressed as

\[
\begin{pmatrix}
  \frac{z + v}{\mu} \\
  -\frac{z}{\mu} + \frac{v}{\tau}
\end{pmatrix}
+ \begin{pmatrix}
  V + a & 0 \\
  0 & -(V - a)
\end{pmatrix}
\begin{pmatrix}
  \frac{z + v}{\mu} \\
  -\frac{z}{\mu} + \frac{v}{\tau}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]

looking at the equivalent system (38) and (39). The information \( z - v/\mu \) travels along characteristics with velocity \( a - V \) in \( \xi-\tau \) space, whereas \( z + v/\mu \) travels along characteristics at velocity \( -a - V \). Recall that \( \mu = \sqrt{\rho} \).

Fig. 2 illustrates the path of characteristic information in (a) the subsonic regime when \( |V| < a_1, a_2 \) and (b) in the supersonic case \( 0 < a_1, a_2 < V \). For the characteristic information from the \( j \)th and \( (j + 1) \)th grid cell at time \( t_n \) to reach the cell interface for times \( t_n \) and \( t_{n+1} \), we must impose the CFL restriction

\[
\max_{i=1,2} \frac{(|V| + a_i)\Delta t}{\Delta x_i} \leq 1.
\]
In the subsonic case, the values of $z_{j+1/2}$ and $v_{j+1/2}$ depend on characteristic information from cells $j$ and $j+1$. For a first order scheme, the data is considered to be constant within each cell taking the value $z_{n}^{j}$ at time $t_{n}$ for $\xi_{j-1/2} < \xi < \xi_{j+1/2}$. At the interfaces, $z$ and $v$ must be continuous, so $z_{j+1/2}$ and $v_{j+1/2}$ as conveyed from cell $j$ must be the same as that conveyed by cell $j+1$, and is the solution of the system

$$
\begin{align}
z_{j+1/2} - \frac{v_{j+1/2}}{\rho_{j}} &= z_{j} - \frac{v_{j}}{\rho_{j}}, \\
v_{j+1/2} - \frac{v_{j+1/2}}{\rho_{j+1}} &= z_{j+1} + \frac{v_{j+1}}{\rho_{j+1}},
\end{align}
$$

(46)

and

$$
\begin{align}
z_{j+1/2} - \frac{v_{j+1/2}}{\rho_{j}} &= z_{j} - \frac{v_{j}}{\rho_{j}}, \\
v_{j+1/2} - \frac{v_{j+1/2}}{\rho_{j+1}} &= z_{j+1} + \frac{v_{j+1}}{\rho_{j+1}},
\end{align}
$$

(47)

However, we find that this scheme is unstable under the usual CFL constraint (45) when $\mu_{1} \neq \mu_{2}$, but quite stable when $\mu_{1} = \mu_{2}$. Recall that $\mu_{i} = \sqrt{\frac{k_{i}}{\rho_{i}}}$.

The reason for the instability may be the same as that for temporal laminates where $V = \infty$. In [9], it is proven that the analytical problem disturbances through a temporal laminate with wavelengths greater than $O(\varepsilon)$ will remain stable, while shorter wave disturbances may become unstable. For numerical computation, it then becomes possible for short wave errors due to round off or discretization to become amplified to such an extent that the integrity of the computation is lost. A new CFL condition is derived for the numerical scheme for the temporal laminate problem that allows for robust computing. Essentially, one must use a $\Delta x$ large enough so that enough numerical diffusion is added to stabilize the computation, and to underresolve the short wave errors. We continue to work on developing a robust numerical algorithm that computes solutions to the supersonic laminate problem.

4. Discussion and validation of effects

In this section, we use the numerical methods given in Section 3 to compute solutions to the heterogeneous problem, and to illustrate the analytical findings of Section 2.

Table 1 shows the values of $\Sigma_{1}$ and $\Gamma_{1}$ corresponding to the heterogeneous spatio-temporal laminate problem when

$$
(\rho_{1}, k_{1}) = (1, 1), \quad (\rho_{2}, k_{2}) = (4, 9), \quad m_{1} = 0.75.
$$

(48)

For these parameters

$$
\frac{1}{\rho(1/\varepsilon)} = 0.96076892, \quad \frac{1}{\rho^{1/2}} = 1.755.
$$
Table 1
Effective values for laminate problem (48)

<table>
<thead>
<tr>
<th>( V )</th>
<th>( \Sigma^+ )</th>
<th>( \Gamma^- )</th>
<th>( \Sigma^- )</th>
<th>( \Gamma^+ )</th>
</tr>
</thead>
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</tr>
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</tr>
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<td>0.00000000</td>
</tr>
<tr>
<td>0.99999999</td>
<td>1.00000033</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
</tr>
</tbody>
</table>

Coordinated wave motion is thus evident for \( 0.9607689 < V < 1 \) and for \( 1.5 < V < 1.75 \), the values of \( \Sigma^+ \) and \( \Sigma^- \) have the same sign. The limit trends in (35)–(37) can also be seen from this data.

In Fig. 3, we show the computed solutions to the heterogeneous spatio-temporal laminate problem with parameters (48) and \( \epsilon = 0.1 \). The initial profile is

\[
z(x, 0) = e^{-2x^2}, \quad v(x, 0) = 0.
\]

This is a direct numerical simulation of the heterogeneous problem as described in the previous section. The wavelength of the initial perturbation can be taken to be about 5. The values of the numerical solution for \( \langle z \rangle \) are obtained by taking the average of consecutive sequences of \( z_n \) covering one \( \epsilon \) period of the laminate, i.e. \( n \) values from material 1 and \( n \) values from material 2, where \( m \Delta x = \epsilon \).

From (27) and Table 1, when \( V = 0.6 \), \( \Gamma^- = 0.0678285 \) and \( \Gamma^+ = 0.00122132 \). In the picture on the right of Fig. 3, we show the results of the computation for this value of \( V \) at time 26; this is the left-going wave. The amplitude of the oscillation due to dispersion is approximately 6% of the amplitude of the dominant wave. The right-going wave exhibits no significant dispersion at this time. The left graph shows the left-going wave in the case \( V = 0 \). We stop the computation at time 158 where the dispersion wave’s height is also 6% of the main wave. Here, \( \Gamma^- = 0.00122112 \). The right-going wave is simply the mirror image of that shown. From (29), the dispersive effects are significant when \( t = O((\Gamma^\pm \epsilon^2)^{-1}) \). Since the ratio of the values of \( \Gamma^- \) for \( V = 0 \) and 0.6 is 6.617, we expect comparable levels of dispersion to be evident in the dynamic laminate 6–7 times earlier as in the static case. This is supported by the results presented.

The exact solution to the convection dispersion equation

\[
u_t + D u_x = bu_{xxx}, \quad u(x, 0) = u_0(x)
\]

is

\[
u(x, t) = \int e^{i(\xi x - \omega t)} d\xi = H(x, 0) * u_0(x),
\]
where $H(x,t)$ is the inverse Fourier transform of $e^{-i a \xi + \frac{5}{3} \xi^3}$. Now,

$$\text{Ai}(\xi) = \int e^{i(\xi x + \xi^3/3)} \, d\xi,$$

so $H(x,t)$ is

$$\int e^{-i \alpha x + \beta x^3} \, e^{i t} \, d\xi = \int e^{i(\xi x - \beta \xi^3)} \, d\xi = \frac{1}{(3\beta)^{1/3}} \text{Ai} \left( \frac{x - at}{(3\beta)^{1/3}} \right) = H(x,t).$$

The solutions to (31) and (32) are therefore

$$\left[ \frac{1}{(3\Sigma x^2)^{1/3}} \text{Ai} \left( \frac{x - \Sigma t}{(3\Sigma x^2)^{1/3}} \right) \right] * z^*_*(s),$$
Fig. 4. Comparable dispersion levels for $V = 0$ and 0.6 using Airy function solution.

and

$$\left[ -\frac{1}{(3\Gamma\varepsilon)^{2/3}} \text{Ai} \left( \frac{x - \Sigma_{\pm}t}{(3\Gamma\varepsilon t)^{1/3}} \right) \right] \ast z_{\pm}(x),$$

where $z_{\pm}$ are the initial data relevant to the averaged waveforms. Fig. 4 computes the exact solution to Eqs. (31) and (32) with parameters (48) and initial data $z_{\pm} = e^{-2x^2/2}$. We see that the exact solutions graphed in this figure compare quite well with the computed solutions in Fig. 3.

In [8], it is shown that the two waveforms travelling at different speeds $\Sigma_{+}$ and $\Sigma_{-}$ will experience different levels of numerical diffusion when $V \neq 0$. When $V = 0$, $a_1 = a_2$ and $\lambda = 1$, we are ensured the least amount of numerical bias. So, in order to further illustrate how well the effective dispersion coefficient represents the homogenized behavior, we use the parameters

$$(k_1, \rho_1) = (1, 1), \quad (k_2, \rho_2) = (10, 10), \quad \varepsilon = 0.1.$$

The numerical solution of the heterogeneous problem experiences significant dispersion at time 300, and plot this result against the solution to the effective equation with the same initial data as before. We use $\Delta x_1 = \Delta x_2 = 0.025$. In order to line up the results, the computed solution is shifted to the right by 0.35; the numerical homogenized
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Fig. 5. Airy solution and computed solution when $V = 0$.

velocity thus differs from that in (30) by $0.35/300 = 0.00117$ which amounts to a 0.2% ‘error’. The dispersed waves align very well (Fig. 5).

5. Conclusion

Understanding the dispersive effects of dynamic materials is very important as we consider the design and implementation of these materials. We have used Floquet analysis to analyze the problem of wave propagation through one-dimensional dynamic laminates. We have derived an effective medium equation that reveals both the hyperbolic and dispersive nature of the medium when the period of the laminate is small relative to the scale of the disturbance. In some regimes, the laminate structures presented here exhibit quite a high level of dispersion for $\varepsilon$ away from 0, so using the second order equation to describe the effective behavior of in those cases the structure will not be adequate for larger times.

For numerical study, natural hyperbolic numerical techniques and stability constraints are no longer applicable in the direct numerical simulation, so further study and sophistication is needed. Numerical diffusion and dispersion also affect the solutions in these heterogeneous problems in ways that still need to be understood further. Knowing
the theoretical solution to the laminate problem helps us to refine our expectation of our numerical results so that we can distinguish numerical artifacts from real physical phenomena.

Acknowledgements

The author acknowledges support of this work through the NSF Grant DMS-0204673.

References

[7] K.A. Lurie, S.L. Weekes, Effective and averaged energy densities in one-dimensional wave propagation through spatio-temporal dielectric laminates with negative effective values of $\epsilon$ and $\mu$, submitted to Nonlinear Analysis and Applications.