# The problem of effective parameters of a mixture of two isotropic dielectrics distributed in space-time and the conservation law for wave impedance in one-dimensional wave propagation

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The problem of effective parameters of a composite medium assembled from the original constituents distributed in space-time is formulated in covariant tensor form. We introduce the 4th rank tensor of material constants using the Maxwell's system for a moving dielectric medium as a model example. For one-dimensional wave propagation, if a mixture is composed from two dielectrics with the same ratio $\epsilon/\mu$ of permittivity $\epsilon$ to permeability $\mu$, then the ratio $\mathcal{E}/M$ of an effective permittivity to an effective permeability of the mixture will preserve the value $\epsilon/\mu$. This statement may be rephrased as the conservation law for the relevant wave impedances $\sqrt{\mu/\epsilon}$; this one is similar to the law known for two-dimensional polycrystals in an analogous elliptic situation.

The tensor concept developed for a dielectric medium is based on the idea of a relativistic invariance of the Maxwell's system. The idea of relativistic invariance is fundamental for an adequate description of the effective parameters of any material assemblage in space-time regardless of its physical embodiment. Problems related to structural vibrations or acoustics could be completely understood on the basis of the relevant relativistic equations.

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## 1. Introduction

The problem indicated in the title has been explored intensively in statics with the original materials distributed in space alone (see Lurie & Cherkaev (1997) and bibliography cited therein). In the present paper, we investigate a similar problem for a *spatio-temporal* distribution of original materials, specifically for a dielectric continuum able to take either the values $(\epsilon_1, \mu_1)$ or the values $(\epsilon_2, \mu_2)$ of its dielectric and magnetic permeabilities at each point $(x, y, z)$ at every instant $t$ of time. This scheme does not necessarily require the physical motion of a material medium to become feasible. The space-time dependent properties can be created through the appropriate "switching" of dielectric constants generated by a suitable electronic scheme. For example, the transmission line may have its linear inductance $L$ and capacitance $C$ take one of the two admissible pairs of values $(L_1, C_1)$ and $(L_2, C_2)$ at each point $(z, t)$ of space-time, the necessary adjustment being implemented due to the electronic (or even mechanical) switching. Observe that this switching may be conducted at *any* speed; particularly , it may be instantaneous. In high frequency electronics, there are known the s.c. "pump waves" used as a means to create a traveling pattern of alternating pairs of parameters along the line (Louisell (1968)). To generate the "wave of linear capacitance", $p-n$ junction diodes may be distributed along the line and appropriately activated in time (Louisell (1968)). The activation of inductance is accomplished through the use of distributed nonlinear elements. The principle of "property wave" allows for a mechanical implementation, too, particularly in acoustics (Morse & Uno Ingard (1968)). The concept of smart materials (Russell (1994)) able to change its properties both in space and time appears to perfectly fit into this scheme as well.

In what follows, we give a mathematical formulation of a problem of mixing of two originally given dielectrics in the framework of Maxwell's theory. This approach is universal in a sense that it can be adjusted, with appropriate modifications, to similar mechanical problems, e.g., the problems of structural vibrations of non-homogeneous elastic bodies. In all circumstances, the material medium may either be assumed immovable or exposed to motion.

In the first case, we apply switching to activate the spatio-temporal material pattern; in the second case, this pattern may be created by the actual material motion.

## 2. The One-dimensional Wave Propagation: A Special Use of Maxwell's System

To simplify technicalities, we begin with the case of a plane electromagnetic wave propagating along the $z$-axis through the medium characterized by isotropic dielectric and magnetic permeabilities $\epsilon(z,t)$ and $\mu(z,t)$. The pair $(\epsilon(z,t), \mu(z,t))$ is assumed to take at each point $(z,t)$ of space-time only one of two admissible pairs of values:

$$(\epsilon(z,t), \mu(z,t)) = \begin{cases} (\epsilon_1, \mu_1) & (\text{"material 1"}), \\ (\epsilon_2, \mu_2) & (\text{"material 2"}). \end{cases} \tag{1}$$

Any relevant region in the $(z,t)$-plane is thus divided into subregions occupied either by material 1 or by material 2. In the absence of currents and charges, the Maxwell's system includes the equations

$$\begin{aligned} \text{curl}\mathbf{E} &= -\mathbf{B}_t, \\ \text{div}\mathbf{B} &= 0, \\ \text{curl}\mathbf{H} &= \mathbf{D}_t, \\ \text{div}\mathbf{D} &= 0. \end{aligned} \tag{2}$$

For a plane electromagnetic wave, this system reduces to

$$\begin{aligned} E_z &= B_t, \\ H_z &= D_t \end{aligned} \tag{3}$$

for the relevant components $E, B, H, D$ of the basic vectors $\mathbf{E}(z,t)$, $\mathbf{B}(z,t)$, $\mathbf{H}(z,t), \mathbf{D}(z,t)$ :

$$\mathbf{E} = E\mathbf{j}, \quad \mathbf{B} = B\mathbf{i}, \quad \mathbf{H} = H\mathbf{i}, \quad \mathbf{D} = D\mathbf{j}. \tag{4}$$

Eqs. (3) stand for the first and the third of Eqs. (2), respectively; the second and the fourth eqs of this system are satisfied identically.

By introducing the scalar potentials $\phi, \phi^*$ and the vector potentials $\mathbf{A}, \mathbf{A}^*$ through the formulas

$$\begin{gathered} \mathbf{E} = -\mathrm{grad}\phi - \mathbf{A}_t, \quad \mathbf{B} = \mathrm{curl}\mathbf{A}, \\ \mathbf{H} = -\mathrm{grad}\phi^* + \mathbf{A}_t^*, \quad \mathbf{D} = \mathrm{curl}\mathbf{A}^*, \end{gathered} \tag{5}$$

and by taking

$$\mathbf{A} = -u\mathbf{j}, \quad \mathbf{A}^* = v\mathbf{i}, \quad \phi = \phi^* = 0, \tag{6}$$

we refer to (4) and obtain

$$E = u_t, \quad B = u_z, \quad H = v_t, \quad D = v_z. \tag{7}$$

Eqs. (3) are thereby satisfied. We now have to apply the material equations to make the Maxwell's system complete. In a laboratory frame (with respect to which a medium is at rest), these equations are given by

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{8}$$

We arrive, finally, at the system

$$v_z = \epsilon u_t, \quad u_z = \mu v_t, \tag{9}$$

governing the wave propagation in one spatial dimension.

We will consider solutions of this system defined in a spatio-temporal region $\Delta$ : $((0,l) \times (0,T))$ and satisfying the boundary and initial conditions

$$u(0,t) = g_1(t), \quad u(l,t) = g_2(t), \tag{10}$$

$$u(z,0) = v(z,0) = 0. \tag{11}$$

The functions $g_i$ will be assumed differentiable infinitely many times, and the compatibility conditions

$$\frac{d^j g_i(0)}{dt^j} = 0 \tag{12}$$

satisfied for all $j \geq 0$.

We will consider smooth solutions of the problem (9)-(11), i.e. the solutions belonging to $W_2^1((0,l) \times (0,T))$.

The smoothness property implies certain restrictions on the shape of the interfaces $L$ between the neighboring subregions of $\Delta$ occupied by different materials. Each interface will be assumed piecewise smooth, i.e. possessing no more than finite number of corner points. Between those points, the slope $V = dz/dt$ of the interface should satisfy the inequality

$$\frac{V^2 - c_1^2}{V^2 - c_2^2} \geq 0 \tag{13}$$

where $c_1^2 = 1/(\epsilon_1\mu_1),\quad c_2^2 = 1/(\epsilon_2\mu_2)$ denote the squares of the velocity of light in materials 1 and 2.

Inequality (13) guarantees the regular transport of the field vectors across the interface $L$ with the observance of the compatibility conditions

$$[u_t + Vu_z]_1^2 = [v_t + Vv_z]_1^2 = 0. \tag{14}$$

To satisfy these conditions, we ultimately need as many as two outgoing characteristics on the interface. Assume that $c_2 > c_1$, then Ineq. (13) will be satisfied either for $V^2 \leq c_1^2$ or for $V^2 \geq c_2^2$. In the first case, there will be one outgoing characteristic pointing to material 1 and another one pointing to material 2 (Fig. 1). In the second case, both characteristics will point into one of those materials (material 2 in Fig. 2). When $c_1 = c_2$, the banned interval $(c_1^2, c_2^2)$ for $V^2$ shrinks to a point. Before we formulate the main problem discussed in this paper with regard to one-dimensional wave propagation, we give a brief account of a general analytic scheme incorporated in electrodynamics of moving media.

### 3. The Maxwell's System for a Moving Dielectric Medium

If a dielectric medium is at rest with respect to the laboratory frame $(x, y, z)$, then in the absence of currents and charges the electromagnetic field in it is described by Eqs. (2) and (8).

The pairs of vectors $\mathbf{B}, \mathbf{E}$ and $\mathbf{H}, \mathbf{D}$ generate, respectively, two skew-symmetric tensors in 4-space $(x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict)$

$$F = (c\mathbf{B}, -i\mathbf{E}) = \begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}, \tag{15}$$

$$f = (\mathbf{H}, -ic\mathbf{D}) = \begin{pmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{pmatrix}. \tag{16}$$

Here, $c$ denotes the velocity $1/\sqrt{\epsilon_0\mu_0}$ of light in a vacuum, and $B_1, \dots, D_3$ are the relevant vector components.

The Maxwell's system (2) now obtains a tensor form incorporated in the equations

$$\frac{\partial F^*_{ik}}{\partial x_k} = 0, \quad \frac{\partial f_{ik}}{\partial x_k} = 0,$$

where $F^*_{ik}$ is a tensor dual to $F_{ik}$, i.e.

$$F^*_{ik} = \frac{1}{2} e_{iklm} F_{lm}.$$

Here, $e_{iklm}$ is a completely antisymmetric tensor of the 4th rank.

## Remark

In a special case (4), (7), we let $F_{1i} = -F_{i1} = F_{34} = -F_{43} = f_{1i} = -f_{i1} = f_{34} = -f_{43} = 0$ for $i = 1, \dots, 4$, and the tensors $F$ and $f$ reduce to

$$F = \begin{pmatrix} 0 & cB & -iE \\ -cB & 0 & 0 \\ iE & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & cu_z & -iu_t \\ -cu_z & 0 & 0 \\ iu_t & 0 & 0 \end{pmatrix} = c \begin{pmatrix} 0 & u_{x_3} & u_{x_4} \\ -u_{x_3} & 0 & 0 \\ -u_{x_4} & 0 & 0 \end{pmatrix}, \tag{17}$$

$$f = \begin{pmatrix} 0 & H & -icD \\ -H & 0 & 0 \\ icD & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v_t & -icv_z \\ -v_t & 0 & 0 \\ icv_z & 0 & 0 \end{pmatrix} = ic \begin{pmatrix} 0 & v_{x_4} & -v_{x_3} \\ -v_{x_4} & 0 & 0 \\ v_{x_3} & 0 & 0 \end{pmatrix}. \tag{18}$$

Because these matrices appear to be the $3 \times 3-$ submatrices of (15) and (16) standing in their right low corners, the numbers $2, 3, 4$ generate the relevant indexation.

The material equations (8) are combined in a single tensor equation - a linear relationship between tensors $f$ and $F$:

$$f = s : F \quad \text{or} \quad f_{ik} = s_{iklm} F_{\ell m}. \tag{19}$$

This equation introduces the fourth rank tensor $s = (s_{iklm})$. This tensor is antisymmetric with respect to indices $i$ and $k$ as well as $l$ and $m$, and symmetric with respect to a simultaneous permutation of indices in the pairs $(i, l)$ and $(k, m)$ as well as to the permutation of pairs $(il)$ and $(km)$:

$$s_{iklm} = -s_{kilm} = -s_{ikml} = s_{lmik} = s_{kiml}. \tag{20}$$

In a laboratory frame, the non-zero components of $s$ are defined for an immovable isotropic dielectric by the formulas:

$$\begin{aligned} s_{1212} &= s_{2121} = -s_{1221} = -s_{2112} = -1/2\mu c, \quad s_{1313} = s_{3131} = -s_{1331} = -s_{3113} = -1/2\mu c, \\ s_{2323} &= s_{3232} = -s_{2332} = -s_{3223} = -1/2\mu c, \\ s_{1414} &= s_{4141} = -s_{1441} = -s_{4114} = -\epsilon c/2, \quad s_{2424} = s_{4242} = -s_{2442} = -s_{4224} = -\epsilon c/2, \\ s_{3434} &= s_{4343} = -s_{3443} = -s_{4334} = -\epsilon c/2. \end{aligned} \tag{21}$$

In the space of skew-symmetric tensors of the 2nd rank in four dimensions we introduce an orthogonal basis defined as a set of skew-symmetric 2nd rank tensors $a_{ik}(i,k=1,2,3)$ specified by the formulas

$$\begin{aligned} a_{12}=1/\sqrt{2}(i_1i_2-i_2i_1),\quad & a_{13}=1/\sqrt{2}(i_1i_3-i_3i_1),\quad & a_{14}=1/\sqrt{2}(i_1i_4-i_4i_1),\\ & a_{23}=1/\sqrt{2}(i_2i_3-i_3i_2),\quad & a_{24}=1/\sqrt{2}(i_2i_4-i_4i_2),\\ & & a_{34}=1/\sqrt{2}(i_3i_4-i_4i_3), \end{aligned} \qquad (22)$$

with

$$a_{ik}\ :\ a^T_{lm}=\begin{cases} 1, & i=l,k=m,\\ 0 & \text{otherwise}\ . \end{cases}$$

In a special case (4), (7), the $F$ - and $f$ - tensors are given by the formulas (see(17), (18) and (7))

$$\begin{aligned} F &= \sqrt{2}(cBa_{23}-iEa_{24})=\sqrt{2}c(u_{x_3}a_{23}+u_{x_4}a_{24}),\\ f &= \sqrt{2}(Ha_{23}-icDa_{24})=\sqrt{2}ic(v_{x_4}a_{23}-v_{x_3}a_{24}). \end{aligned} \qquad (23)$$

Getting back to a general case, let us consider tensors $a_{ik}$ of the set (22) as the primary entities and introduce the elementary symmetric functions of the second degree of these tensors. We obtain as many as 21 such functions given by the following table:

$$\begin{aligned} &a_{12}a_{12}, a_{12}a_{13}+a_{13}a_{12}, a_{12}a_{14}+a_{14}a_{12}, a_{12}a_{23}+a_{23}a_{12}, a_{12}a_{24}+a_{24}a_{12}, a_{12}a_{34}+a_{34}a_{12},\\ &a_{13}a_{13}, a_{13}a_{14}+a_{14}a_{13}, a_{13}a_{23}+a_{23}a_{13}, a_{13}a_{24}+a_{24}a_{13}, a_{13}a_{34}+a_{34}a_{13},\\ &a_{14}a_{14}, a_{14}a_{23}+a_{23}a_{14}, a_{14}a_{24}+a_{24}a_{14}, a_{14}a_{34}+a_{34}a_{14}, \qquad (24)\\ &a_{23}a_{23}, a_{23}a_{24}+a_{24}a_{23}, a_{23}a_{34}+a_{34}a_{23},\\ &a_{24}a_{24}, a_{24}a_{34}+a_{34}a_{24},\\ &a_{34}a_{34}. \end{aligned}$$

The most general linear form of these functions represents the second rank symmetric tensor in the space of $a_{ik}$, i.e. the fourth rank tensor in the original space $(x_1,x_2,x_3,x_4)$. This latter tensor possesses a special symmetry of indices given by formulas (20).

A unit tensor $e$ in the space of skew-symmetric 2nd rank tensors $a_{ik}$ is given by

$$e=-a_{12}a_{12}-a_{13}a_{13}-a_{14}a_{14}-a_{23}a_{23}-a_{24}a_{24}-a_{34}a_{34}. \qquad (25)$$

The $s$-tensor for an isotropic dielectric medium that does not move with respect to a laboratory frame is given in this frame by the formula

$$s=-\frac{1}{\mu c}(a_{12}a_{12}+a_{13}a_{13}+a_{23}a_{23})-\epsilon c(a_{14}a_{14}+a_{24}a_{24}+a_{34}a_{34}). \qquad (26)$$

Eq. (19) is a tensor equation between two tensors $F$ and $f$ (see (15), (16)) in a Euclidean 4-space $(x_1, x_2, x_3, x_4)$. The proportionality factor $s$ in this equation is characterized by a clear geometric structure in the case of isotropic dielectric. This structure is incorporated in Eqs. (26) and (22). We observe that a purely spatial rotation does not affect either of the tensors $a_{12}a_{12} + a_{13}a_{13} + a_{23}a_{23}$ and $a_{14}a_{14} + a_{24}a_{24} + a_{34}a_{34}$ appearing in parentheses at the rhs of (26). A uniform motion of the coordinate frame with the velocity $V$ along the $x_3$-axis is equivalent to its rotation about the $(x_1, x_2)$-plane by an imaginary angle $i\psi$ in the 4-space, with the value of $\psi$ being completely determined by $V$. In a moving ("primed") frame, the components of $F$ and $f$ may be expressed through their values in a laboratory frame by virtue of the formulas expressing the effect of rotation (Lorentz transformation). This procedure will affect all of the tensor dyads in (26) except $a_{12}a_{12}$ and $a_{34}a_{34}$ generating the well-known Minkowski's material relations for a moving medium. The $s$-tensor (26) is thus isotropic with respect to purely spatial rotations and anisotropic with respect to the motion of the coordinate frame. It will become *completely isotropic*, i.e. isotropic with respect to all rotations in 4-space, provided that $1/(\mu c) = \epsilon c$, i.e. $c^2 = 1/(\epsilon\mu)$. This case is exceptional: it holds for a vacuum where $\epsilon = \epsilon_0$, $\mu = \mu_0$, $c^2 = 1/(\epsilon_0\mu_0)$, and the $s$-tensor becomes proportional to a unit tensor (25)

$$s = \sqrt{\frac{\epsilon_0}{\mu_0}} e.$$

Disregarding this case, we may single out the following problems of mixing the materials isotropic in a conventional sense. We first consider the spatio-temporal microstructures generated by two different isotropic materials occupying periodic cells in space-time but *motionless in a laboratory frame*. This means that the relevant tensors $s$ differ only in their eigenvalues, their eigentensors $a_{ik}$ being preserved identical. This problem belongs to the type mentioned in the introduction. Another thinkable problem may be that of a "spatio-temporal polycrystal" in which there appear microstructures assembled from fragments of one and the same isotropic material brought each to *its own motion*. This pattern represents a direct analog of a polycrystal assembled from fragments that differ only by their orientation with regard to laboratory axes. In space-time, the difference in orientation is due to a relative motion.

This motion may be arranged as a high frequency background mechanical vibration of a dielectric continuum in the form of a standing wave.

We now apply the covariant formulation to show that the effective properties of the spatio-temporal microstructures allow for an important detailization in a special case of one-dimensional wave propagation. Specifically, we assume that the $s$-tensors of participating materials have the same determinant, i.e. the product of their eigenvalues $d_1 d_2$, or the same ratio $\epsilon/\mu$. In these circumstances, the determinant det $s_0$ of the effective tensor $s_0$ preserves the same value.

The ratio $\mu/\epsilon$ is known to be equal to $\rho^2$ where $\rho$ denotes the wave impedance of a dielectric medium. We thus arrive at the *conservation law of the wave impedance* of a spatio-temporal mixture of two dielectrics possessing equal wave impedances and exposed to one-dimensional wave motion.

## 4. Conservation of the Wave Impedance Through One-dimensional Wave Propagation

Because of (23), we will be interested in rotation of a coordinate frame $(x_1, x_2, x_3, x_4)$ that leaves intact the $x_1$ and $x_2$-axes. This rotation introduces a primed coordinate frame; it affects the vectors $i_3, i_4$ and consequently, all of the tensors $a_{ik}$ listed in (22) except $a_{12}$ and $a_{34}$. The vectors $i'_3$ and $i'_4$ will become linear functions of $i_3$ and $i_4$, and tensors $a'_{13}$ and $a'_{14}$ will consequently become linear functions of $a_{13}$ and $a_{14}$, and tensors $a'_{23}, a'_{24}$ will depend on $a_{23}$ and $a_{24}$.

The described rotation occurs in a subspace of tensors $a_{23}, a_{24}$; the unit tensor in this subspace becomes

$$e = -a_{23}a_{23} - a_{24}a_{24},$$

whereas the $s$-tensor in it may be specified as

$$s = -\frac{1}{\mu c}a_{23}a_{23} - \epsilon c a_{24}a_{24}. \tag{27}$$

Consider the tensor

$$O = a_{23}a_{24} - a_{24}a_{23} \tag{28}$$

and introduce two linearly independent tensors $F(1)$ and $F(2)$ (see (23)) generated, respectively, by two linearly independent test fields $u(1)$ and $u(2)$. The expression

$$F(1) \;:\; O \;:\; F(2) = 2c^2(u_{x_3}(1)u_{x_4}(2) - u_{x_4}(1)u_{x_3}(2)) = 2c^2 \det(\nabla u(1), \nabla u(2))$$

is quasiaffine in $W_2^1(\Delta)$ (Lurie & Cherkaev (1997)), i.e.

$$\lim \mathrm{wk}_{L_1(\Delta)} 2c^2 \det(\nabla u^{(r)}(1), \nabla u^{(r)}(2)) = 2c^2 \det(\nabla u^{(0)}(1), \nabla u^{(0)}(2)), \tag{29}$$

where $r$ is the index of partition of $\Delta$ into subdomains occupied by materials 1 or 2, and

$$\lim \mathrm{wk}_{W_2^1(\Delta)} u^{(r)}(i) = u^{(0)}(i),\ i = 1, 2.$$

A similar behavior is demonstrated by the expression

$$f(2) \;:\; O \;:\; f(1) = 2c^2(v_{x_3}(1)v_{x_4}(2) - v_{x_4}(1)v_{x_3}(2)) = 2c^2 \det(\nabla v(1), \nabla v(2)).$$

This one is also quasiaffine, i.e.

$$\lim \mathrm{wk}_{L_1(\Delta)} 2c^2 \det(\nabla v^{(r)}(1), \nabla v^{(r)}(2)) = 2c^2 \det(\nabla v^{(0)}(1), \nabla v^{(0)}(2)) \tag{30}$$

where

$$\lim \mathrm{wk}_{W_2^1(\Delta)} v^{(r)}(i) = v^{(0)}(i),\ i = 1, 2.$$

Because $f = s : F$, we refer to (23) and rewrite (30) as

$$\lim \mathrm{wk}_{L_1(\Delta)}(\det s^{(r)}) \det(\nabla u^{(r)}(1), \nabla u^{(r)}(2)) = (\det s_0) \det(\nabla u^{(0)}(1), \nabla u^{(0)}(2)). \quad (31)$$

Assume now that

$$\det s^{(r)} = \epsilon/\mu = \mathrm{const}(r);$$

then, comparing (31) and (29), we conclude that

$$\det s_0 = \epsilon/\mu. \quad (32)$$

In other words, if the determinant of the $s$-tensor (27) is the same for both of the original materials (1), then the effective tensor $s_0$ possesses the same determinant. This is a hyperbolic analog of the conservation law that holds for a two-dimensional polycrystal appearing in planar elliptic problems of the 2nd order as a two-dimensional composite microstructure. We see that a similar law also holds for spatio-temporal composites in one spatial dimension; the microgeometry of such composites in space-time should, however, be consistent with Ineq. (13). An example of such a microgeometry is given by a chess-board pattern shown in Fig. 3.

The above proof assumes that the material tensors $s^{(r)}$ in the subdomains remain isotropic with respect to spatial rotations and possess the same value of $\det s^{(r)}$; otherwise, these tensors may be arbitrary. This means, in particular, that $s^{(r)}$ may possess different eigentensors $a_{23}^{(r)}$, $a_{24}^{(r)}$, in other words, isotropic materials occupying different subdomains may participate each in its own independent motion. The conservation law (32) therefore holds true for both types of the mixing problems discussed in Section 3.

In Appendix, we confirm Eq. (32) for two selected mixtures of the first type; specifically, we consider spatio-temporal laminates of the first rank (Appendix A), and of the second rank (Appendix B).

## Appendix A: Spatio-Temporal Laminates of Rank 1

The effective parameters of a simple spatio-temporal laminate in one spatial dimension were computed in Lurie (1997). Using the notation (1), we consider Eqs. (9) in which the material parameters $\epsilon, \mu$ are assumed dependent on the argument $(z - Vt)/\delta$ where $\delta$ denotes a small parameter. This dependence is specified by the following assumptions:

1. the characterization (1) holds;

2. materials 1 and 2 are placed within the alternating layers in the $(z, t)$-plane, these layers occupying, respectively, the $m$th and (1-$m$)th part of the period of a microstructure. The slope $V = dz/dt$ of the layers is so chosen as to ensure the observance of Ineq. (13).

After homogenization, the system (9) is replaced by

$$\begin{aligned} \alpha u_z &+ \beta V u_t = V v_z + v_t, \\ V u_z &+ u_t = \theta(\alpha v_z + \beta V v_t), \end{aligned} \quad (A.1)$$

with parameters $\alpha, \beta, \theta$ defined as (Lurie (1997))

$$
\begin{aligned}
\alpha &= \frac{1}{\mu_1\mu_2}\frac{\overline{\left(\frac{1}{c^2}\right)}V^2 - 1}{\bar{\epsilon}V^2 - \overline{\left(\frac{1}{\mu}\right)}}, \\
\beta &= \epsilon_1\epsilon_2\frac{V^2 - \bar{c^2}}{\bar{\epsilon}V^2 - \overline{\left(\frac{1}{\mu}\right)}}, \\
\theta &= \frac{\bar{\epsilon}V^2 - \overline{\left(\frac{1}{\mu}\right)}}{\epsilon_1\epsilon_2\langle\frac{1}{\mu}\rangle V^2 - \frac{1}{\mu_1\mu_2}\langle\epsilon\rangle},
\end{aligned}
\tag{A.2}
$$

$c_1 = 1/\sqrt{\epsilon_1\mu_1}$, $\bar{\epsilon} = m\epsilon_2 + (1-m)\epsilon_1$, $\langle\epsilon\rangle = m\epsilon_1 + (1-m)\epsilon_2$, etc. In (A.1), we preserve the original symbols $u, v$ to denote the weak limits of these quantities, i.e. their values averaged over the cell of periodicity.

An equivalent system of equations is given by the formulas

$$
\begin{aligned}
pu_z - qu_t &= v_t, \\
qu_z + ru_t &= v_z,
\end{aligned}
\tag{A.3}
$$

with parameters $p, q, r$ defined as

$$
p = \frac{V^2 - \theta\alpha^2}{\theta(V^2\beta - \alpha)}, \quad q = \frac{V(1-\theta\alpha\beta)}{\theta(V^2\beta - \alpha)}, \quad r = -\frac{1 - V^2\theta\beta^2}{\theta(V^2\beta - \alpha)}. \tag{A.4}
$$

The system (A.3) can be simplified if we introduce the "primed" coordinate frame $z', t'$ moving with a suitable velocity $w$ with respect to the laboratory frame $z, t$. Coordinates $z', t'$ are linked with $z, t$ by the Lorentz transform

$$
z' = \gamma^{-1}(z - wt), \quad t' = \gamma^{-1}(t - \frac{w}{c^2}z), \quad \gamma = \sqrt{1 - w^2/c^2}.
$$

Eqs. (A.3) now obtain the form

$$
\begin{aligned}
&(p + 2qw - rw^2)u_{z'} - \left[q + \left(\frac{p}{c^2} - r\right)w + q\frac{w^2}{c^2}\right]u_{t'} = \gamma^2 v_{t'}, \\
&\left[q + \left(\frac{p}{c^2} - r\right)w + q\frac{w^2}{c^2}\right]u_{z'} + \left[r - 2q\frac{w}{c^2} - p\frac{w^2}{c^4}\right]u_{t'} = \gamma^2 v_{z'}.
\end{aligned}
\tag{A.5}
$$

Define $w$ by the equation

$$
\frac{q}{c^2}w^2 + \left(\frac{p}{c^2} - r\right)w + q = 0, \tag{A.6}
$$

i.e. as

$$
w = \frac{c^2}{2q}\left[-\frac{p}{c^2} + r - \sqrt{\left(\frac{p}{c^2} - r\right)^2 - \frac{4q^2}{c^2}}\right]; \tag{A.7}
$$

then, by eliminating $r$ from (A.5) with the aid of (A.6), we reduce (A.5) to the form

$$(p+qw)u_{z'} = v_{t'},$$
$$\left(\frac{p}{c^2}+\frac{q}{w}\right)u_{t'} = v_{z'}. \qquad \text{(A.8)}$$

In a primed frame (cf. Eq. (9)), the complex $p+qw$ is treated as an inverse magnetic permeability $1/M$, and the factor $p/c^2+q/w$ appears to be an effective dielectric permeability $\mathcal{E}$.

Referring to (A.7), we show by direct inspection that

$$\mathcal{E}/M = \left(\frac{p}{c^2}+\frac{q}{w}\right)(p+qw) = pr+q^2. \qquad \text{(A.9)}$$

Eqs. (A.4) now yield

$$pr+q^2 = 1/\theta, \qquad \text{(A.10)}$$

and, since

$$\epsilon_1/\mu_1 = \epsilon_2/\mu_2 = g, \qquad \text{(A.11)}$$

the third Eq. (A.2) shows that

$$\frac{1}{\theta} = g, \qquad \text{(A.12)}$$

which, together with (A.9) and (A.10), confirms the desired result:

$$\mathcal{E}/M = g.$$

## Appendix B: Modulation: Spatio-Temporal Laminates of Rank 2

We consider a rank two spatio-temporal laminate assembled from rank one composites $C1$ and $C2$ used as original materials instead of materials 1 and 2 above. These composites are in their turn laminates constructed from materials 1 and 2 applied on a smaller scale.

Composites $C1$ and $C2$ differ from each other by the values of $m, V$, and, consequently, by the corresponding phase velocities $\wedge_{1i}$ and $\wedge_{2i}$, $i=1,2$ (Lurie(1997)). The relevant values for $C1$ will be denoted as $m_{11}, V_1, \wedge_{11}, \wedge_{12}$, and for $C2$ as $m_{21}, V_2, \wedge_{21}, \wedge_{22}$.

The rank two composite will be characterized by the volume fraction $m_1$ of $C1$ in it, as well as by the slope $U$ of alternating layers filled by $C1$ and $C2$ (Fig. 4). The overall volume fraction $m$ of material 1 in the rank two composite will be equal to

$$m = m_1 m_{11} + (1-m_1)m_{21},$$

and material 2 in this composite will be represented in a volume fraction

$$1-m = m_1(1-m_{11}) + (1-m_1)(1-m_{21}).$$

The slopes $V_1, V_2$ should both satisfy Ineq. (13), and the slope $U$ must obey the inequality

$$\frac{U-\Lambda_{11})(U-\Lambda_{12})}{(U-\Lambda_{21})(U-\Lambda_{22})}\geq 0$$

in order to observe compatibility conditions (14) on the interface between the layers occupied by $C1$ and $C2$. The effective behavior of the rank two composite will be given by the system

$$\begin{aligned} Au_z &+ BUu_t = Uv_z + v_t, \\ Uu_z &+ u_t = \Theta(Av_z + BUv_t), \end{aligned} \tag{B.13}$$

similar to (A.1).

As before, we preserve symbols $u$ and $v$ to designate the weak limits of these quantities, the symbols $A, B, \Theta$ are defined by

$$A=\frac{\langle\frac{\pi}{D}\rangle}{\langle\frac{1}{D}\rangle},\ \ BU=\frac{\langle\frac{\rho}{D}\rangle}{\langle\frac{1}{D}\rangle},\ \ \Theta=\frac{\langle\frac{1}{D}\rangle}{\langle\frac{pr+q^2}{D}\rangle}, \tag{B.14}$$

where

$$\pi = p+qU,\ \ \rho = rU-q,\ \ D=\rho U-\pi. \tag{B.15}$$

Parameters $p, q, r$ in these formulas are defined for each of the components $C1$ and $C2$ by the relevant Eqs. (A.4); we apply notation $p_1, q_1, r_1$ and $p_2, q_2, r_2$ for these parameters, as well as the concurrent notation $\alpha_1, \beta_1, \theta_1, V_1$ and $\alpha_2, \beta_2, \theta_2, V_2$. The operation $\langle\cdot\rangle$ in (B.14) is defined as

$$\langle\cdot\rangle = m_1(\cdot)_1 + (1-m_1)(\cdot)_2.$$

After some calculation we obtain the formulas (cf. (A.2))

$$\begin{aligned} A &= \frac{\pi_1\pi_2\left[U\overline{\left(\frac{\rho}{\pi}\right)}-1\right]}{U\bar{\rho}-\bar{\pi}}, \\ BU &= \frac{\rho_1\rho_2\left[U-\overline{\left(\frac{\pi}{\rho}\right)}\right]}{U\bar{\rho}-\bar{\pi}}, \\ \Theta &= \frac{U\bar{\rho}-\bar{\pi}}{\kappa}, \end{aligned} \tag{B.16}$$

$$\kappa = \rho_1\rho_2\langle p\rangle - \pi_1\pi_2\langle r\rangle + \langle q\rangle(\rho_1\pi_2+\rho_2\pi_1).$$

Here, again, $\pi_1 = p_1 + q_1 U$, $\langle p\rangle = m_1 p_1 + (1-m_1)p_2$, $\bar{\pi} = m_1\pi_2 + (1-m_1)\pi_1$, etc. We now introduce the quantities $P, Q, R$, linked with $A, B, \Theta, U$ by the formulas (A.4) in which $P, Q, R$ replace $p, q, r$, and $A, B, \Theta, U$ replace $\alpha, \beta, \theta, V$. The system (B.13) is equivalent to

$$\begin{aligned} Pu_z &- Qu_t = v_t, \\ Qu_z &+ Ru_t = v_z, \end{aligned} \tag{B.17}$$

(an analog of (A.3)). We introduce new coordinates $z', t'$ by the Lorentz transform

$$z' = \Gamma^{-1}(z - Wt), \;\; t' = \Gamma^{-1}(t - \frac{W}{c^2}), \;\; \Gamma = \sqrt{1 - W^2/c^2},$$

where $W$ is defined by (cf. (A.6) and (A.7))

$$\frac{Q}{c^2}W^2 + \left(\frac{P}{c^2} - R\right) W + Q = 0,$$

$$W = \frac{c^2}{2Q}\left[-\frac{P}{c^2} + R - \sqrt{\left(\frac{P}{c^2} - R\right)^2 - \frac{4Q^2}{c^2}}\right].$$

Eqs. (B.17) now become (cf. (A.8))

$$(P + QW)u_{z'} = v_{t'},$$

$$\left(\frac{P}{c^2} + \frac{Q}{W}\right) u_{t'} = v_{z'}.$$

As before, we interpret $P + QW$ as $1/M$, and $P/c^2 + Q/W$ as $\mathcal{E}$, and show that

$$\mathcal{E}/M = \left(\frac{P}{c^2} + \frac{Q}{W}\right)(P + QW) = PR + Q^2 = 1/\Theta.$$

Referring to (A.11), we will now prove that

$$\frac{1}{\Theta} = g.$$

To demonstrate this, apply direct calculation to show that (see (B.15), (B.16))

$$U\bar{\rho} - \bar{\pi} = m_1(U^2 r_2 - 2Uq_2 - p_2) + m_2(U^2 r_1 - 2Uq_1 - p_1) = m_1 S_2 + m_2 S_1 = \bar{S},$$

and

$$\kappa = m_1(p_1 r_1 + q_1^2)S_2 + m_2(p_2 r_2 + q_2^2)S_1.$$

We now have by (A.10)-(A.12)

$$p_1 r_1 + q_1^2 = \frac{1}{\theta_1} = p_2 r_2 + q_2^2 = \frac{1}{\theta_2} = \frac{\epsilon_1}{\mu_1} = \frac{\epsilon_2}{\mu_2} = g,$$

and, consequently,

$$\frac{1}{\Theta} = \frac{\kappa}{U\bar{\rho} - \bar{\pi}} = \frac{(p_1 r_1 + q_1^2)\bar{S}}{\bar{S}} = p_1 r_1 + q_1^2 = g, \;\; QED.$$

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## List of Captions

Fig. 1 Case $V^2 < c_1^2$.

Fig. 2 Case $V^2 > c_2^2$.

Fig. 3 An assemblage for which (32) holds.

Fig. 4 Rank two composite in space-time.

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