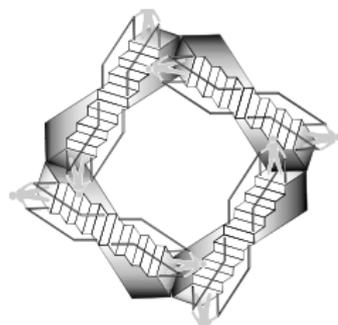


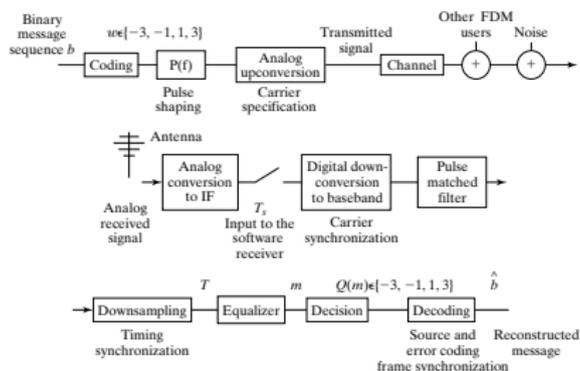
PULSE SHAPING AND RECEIVE FILTERING

- ★ Pulse and Pulse Amplitude Modulated Message Spectrum
- ★ Eye Diagram
- ★ Nyquist Pulses
- ★ Matched Filtering
- ★ Matched, Nyquist Transmit and Receive Filter Combination

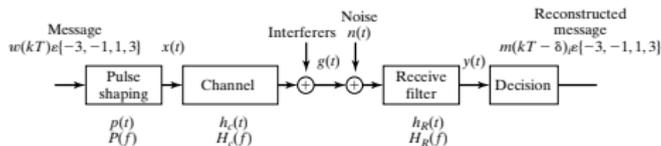


adaptive components

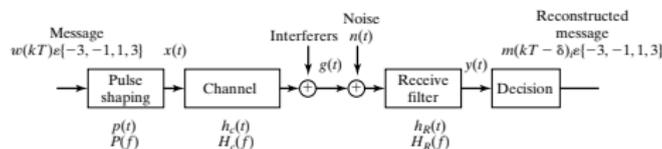
Pulse Shaping and Receive Filtering



We will focus on the situation where up and downconversion have been flawlessly performed and the effect of transmission from baseband PAM message waveform to received signal is presumed described by a linear transfer function and the addition of interferers, in particular spectrally flat broadband noise.



Pulse and pulse amplitude modulated (PAM) message spectrum



The spectral footprint of a baseband PAM signal is no wider than that of the pulse shape.

- ▶ Compose the analog pulse train entering the pulse shaping filter as

$$w_a(t) = \sum_k w(kT)\delta(t - kT)$$

which is $w(kT)$ for $t = kT$ and 0 for $t \neq kT$

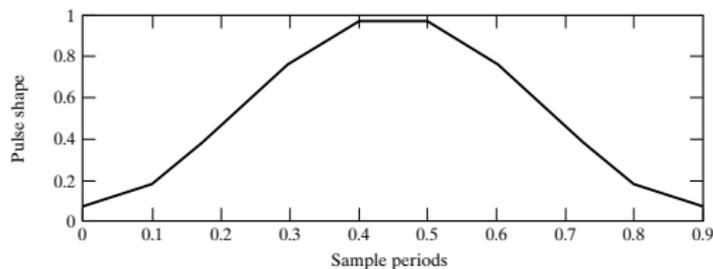
- ▶ Pulse shaping filter output

$$x(t) = w_a(t) * p(t) \Rightarrow X(f) = W_a(f)P(f)$$

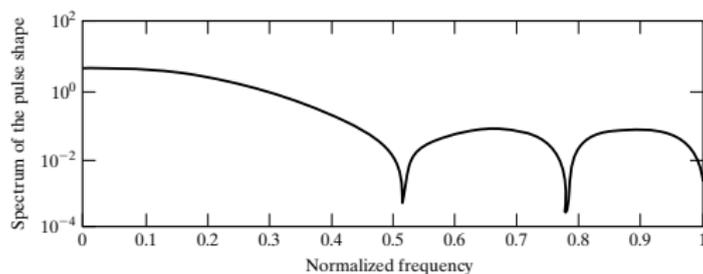
- ▶ $X(f)$ cannot be nonzero at frequencies where $P(f)$ is zero.

Pulse ... message spectrum (cont'd)

One-symbol wide Hamming blip pulse shape (with 10 samples per symbol) and frequency response (from freqz)



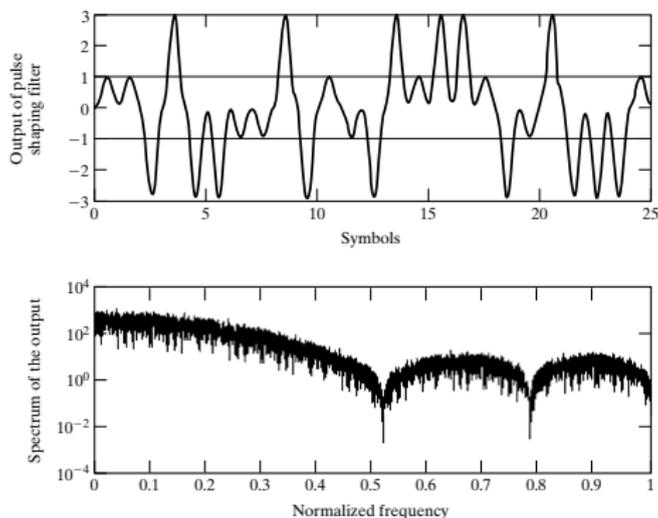
(a)



(b)

Pulse ... message spectrum (cont'd)

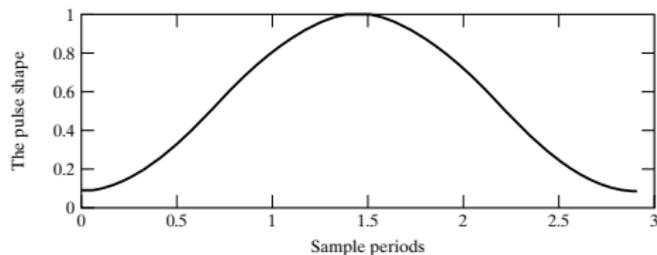
Spectrally flat 4-PAM symbol sequence triggering baud-spaced 10-times oversampled Hamming blip pulse shape as (baseband) output of pulse shaping filter



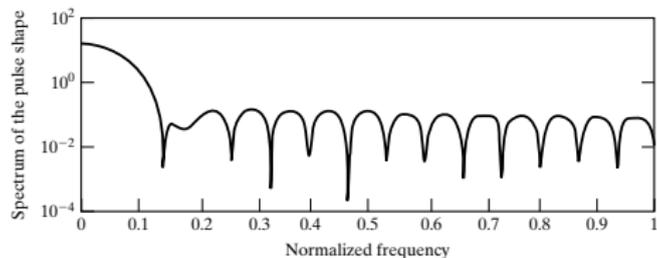
Message signal spectrum has scalloped contours of Hamming blip pulse frequency response.

Pulse ... message spectrum (cont'd)

Triple-wide Hamming blip



(a)

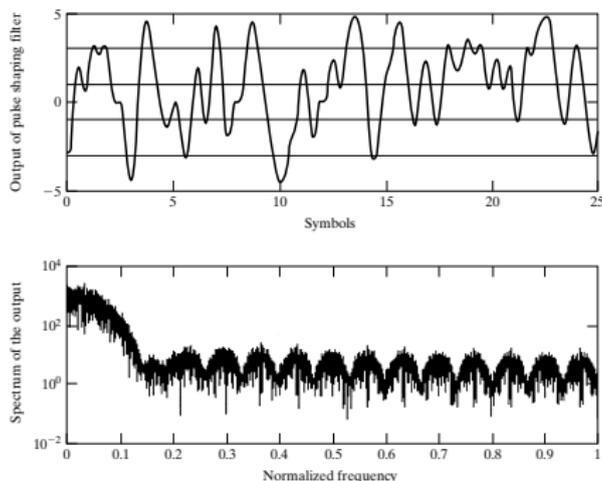


(b)

Wider pulse shape \Rightarrow narrower passband in magnitude spectrum

Pulse ... message spectrum (cont'd)

Spectrally flat 4-PAM symbol sequence triggering three-baud-wide 10-times oversampled Hamming blip pulse shape as (baseband) output of pulse shaping filter

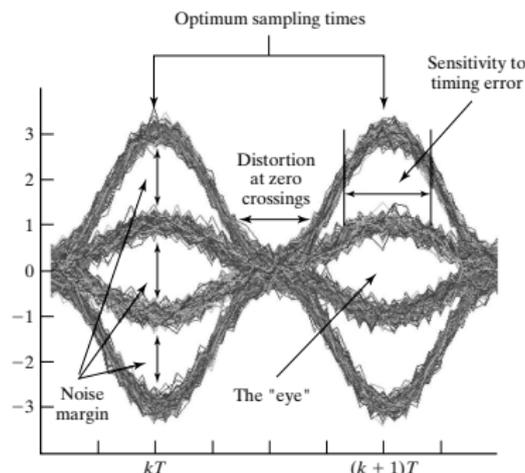


Compare message with single-baud-wide Hamming pulse to observe how intersymbol interference of triple-baud-wide Hamming pulse can cause decision errors.

Eye Diagram

Eye diagram is a popular robustness evaluation tool.

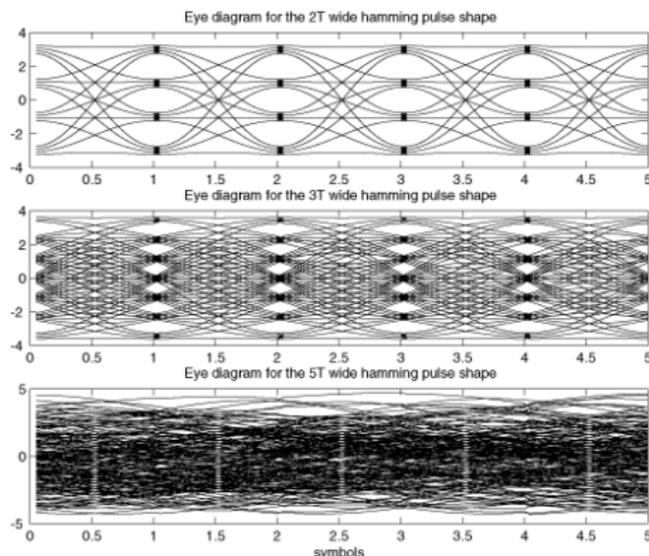
For 4-PAM, single-baud-wide Hamming blip with additive broadband channel noise, retriggering oscilloscope after every 2 baud intervals produces



Observe illustrative vertical (amplitude) and horizontal (timing) margins for correct decision at sample times.

Eye Diagram (cont'd)

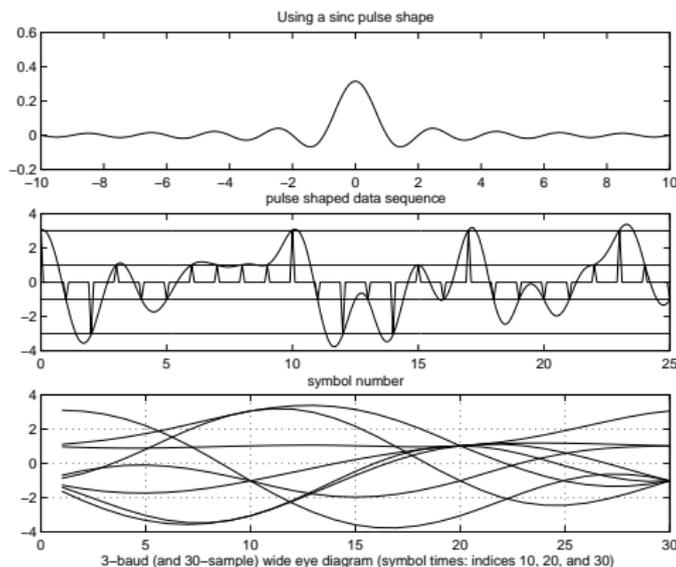
Reconsider multiple-baud-wide Hamming pulse example.



- ▶ Top: Double → open-eye
- ▶ Middle: Triple → partial eye closure
- ▶ Bottom: Quintuple → fully closed eye

Eye Diagram (cont'd)

Consider 20-symbol wide, 10 times oversampled, truncated, sinc pulse ($\sin(\pi t/T)/(\pi t/T)$) with zero-crossings at kT for $k = 1, 2, \dots, 10$) for 4-PAM symbol sequence



A multi-baud-wide pulse shape, but no ISI!

Nyquist Pulses

The impulse response of a Nyquist pulse creating no ISI at other sample times is zero at those instants and nonzero only at the one particular sample time.

- ▶ The impulse response $p(t)$ is a Nyquist pulse for a T -spaced symbol sequence if there exists a τ such that

$$p(t)|_{t=kT+\tau} = \begin{cases} c, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

- ▶ Rectangular pulse:

$$p_R(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

- ◉ Rectangular pulse is a Nyquist pulse.

Nyquist Pulses (cont'd)

- ▶ Sinc pulse:

$$p_S(t) = \frac{\sin \pi f_0 t}{\pi f_0 t}$$

where $f_0 = 1/T$.

- ⊙ Sinc is Nyquist pulse because $p_S(0) = 1$ and $p_S(kT) = \frac{\sin(\pi k)}{\pi k} = 0$.
 - ⊙ Sinc envelope decays as $1/t$.
- ▶ Raised-cosine pulse:

$$p_{RC}(t) = 2f_0 \left(\frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \right) \left[\frac{\cos(2\pi f_\Delta t)}{1 - (4f_\Delta t)^2} \right]$$

with roll-off factor $\beta = f_\Delta/f_0$.

- ⊙ Raised-cosine is Nyquist pulse for $T = 1/2f_0$ because p_{RC} has a sinc factor $\sin(\pi k)/\pi k$ which is zero for all nonzero integers k .
 - ⊙ Raised-cosine envelope decays at $1/|t^3|$.
 - ⊙ As $\beta \rightarrow 0$, raised-cosine \rightarrow sinc.

Nyquist Pulses (cont'd)

- ▶ Raised-cosine pulse (cont'd):
 - ⊙ Fourier transform

$$P_{RC}(f) = \begin{cases} 1, & |f| < f_1 \\ \frac{1+\cos(\alpha)}{2}, & f_1 < |f| < B \\ 0, & |f| > B \end{cases}$$

where

B is the absolute bandwidth,

f_0 is the 6db bandwidth,

$f_{\Delta} = B - f_0$,

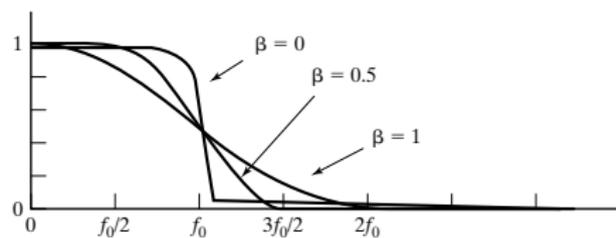
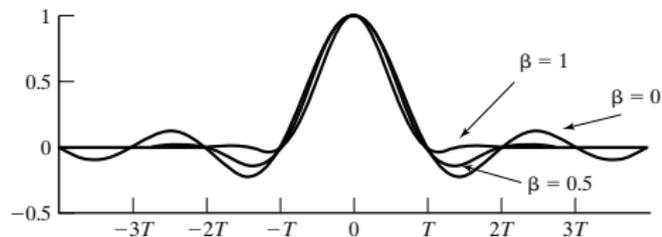
$f_1 = f_0 - f_{\Delta}$, and

$\alpha = \frac{\pi(|f| - f_1)}{2f_{\Delta}}$

Nyquist Pulses (cont'd)

► Raised-cosine pulse (cont'd):

⊙ Time and Frequency Plots:



Nyquist pulses (cont'd)

The sum of the frequency responses of a Nyquist pulse shape and its replicas shifted by an integer multiple of the symbol frequency (i.e. the inverse of the symbol period) is a real constant.

- ▶ Consider a candidate Nyquist pulse $v(t)$ that is nonzero at time zero and zero for all other times that are integer multiples of the symbol period T .
- ▶ Using the sifting property of (A.56) rewritten with frequency as the independent variable and utilizing the fact that δ is an even function as

$$V(f) * \delta(f - f_0) = V(f - f_0)$$

yields

$$\sum_{n=-\infty}^{\infty} V(f - nf_0) = V(f) * \left[\sum_{n=-\infty}^{\infty} \delta(f - nf_0) \right]$$

Nyquist pulses (cont'd)

- ▶ From (A.28) with $w(t) = 1$ and $f_0 = 1/T$

$$\mathcal{F}\left\{T \sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} = \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

- ▶ Given (A.15) and (A.39)

$$\begin{aligned} V(f) * \left[\sum_{n=-\infty}^{\infty} \delta(f - nf_0)\right] \\ &= \int_{t=-\infty}^{\infty} [v(t)(T \sum_{k=-\infty}^{\infty} \delta(t - kT))] e^{-j2\pi ft} dt \\ &= \sum_{k=-\infty}^{\infty} T v(kT) e^{-j2\pi f kT} \end{aligned}$$

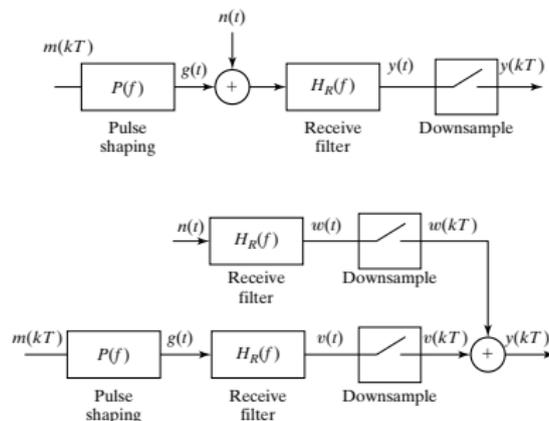
- ▶ Because $v(kT)$ is nonzero only for $v(0)$,

$$\sum_{n=-\infty}^{\infty} V(f - nf_0) = T v(0)$$

- ▶ Thus, sum of $V(f - nf_0)$ is a real constant if $v(t)$ is a Nyquist pulse. Converse is also true.

Matched Filter

Suppose the channel simply adds broadband noise $n(t)$. The symbol to reconstructed downsample system is described by



so $y(t) = v(t) + w(t) = h_R(t) * g(t) + h_R(t) * n(t)$.

- Our objective is to choose $h_R(t)$ to maximize the power of the signal $v(t)$ at a specific time $t = \tau$, i.e. $v^2(\tau)$, relative to the total power of $w(t)$ where the power spectral density of $n(t)$ is a constant η over all frequencies.

Matched Filter (cont'd)

With spectrally flat channel noise the SNR-maximizing receive filter impulse response is the time-reversal of that of the pulse shape.

- ▶ The Fourier transform of the autocorrelation function of $w(t)$

$$R_w(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} w(t)w(t + \tau)dt$$

equals the power spectral density of $w(t)$

$$\mathcal{P}_w(f) = \lim_{T \rightarrow \infty} \frac{|W_T(f)|^2}{T}$$

where $W_T(f)$ is the Fourier transform of the truncated $w(t)$

$$w_T(t) = \begin{cases} w(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

Matched Filter (cont'd)

- ▶ From (E.4), the power spectral density of the output y of a linear filter h with input u is

$$\mathcal{P}_y(f) = \mathcal{P}_u(f)|H(f)|^2$$

- ▶ Thus, with noise n having a flat power spectral density

$$\mathcal{P}_w(f) = \mathcal{P}_n(f)|H_R(f)|^2 = \eta|H_R(f)|^2$$

- ▶ From (E.2), total power in w is

$$P_w = \int_{f=-\infty}^{\infty} \mathcal{P}_w(f)df$$

- ▶ With our objective of choosing $h_R(t)$ to maximize the power of the signal $v(t)$ at a specific time $t = \tau$, i.e. $v^2(\tau)$, relative to the total power of $w(t)$, we now need to compute $v^2(\tau)$.

Matched Filter (cont'd)

- ▶ Turning to calculation of v^2 , from (A.16)

$$v(\tau) = \int_{f=-\infty}^{\infty} V(f)e^{j2\pi f\tau} df$$

where $V(f) = H_R(f)G(f)$, so

$$v^2(\tau) = \left| \int_{-\infty}^{\infty} H_R(f)G(f)e^{j2\pi f\tau} df \right|^2$$

- ▶ The quantity to be maximized is

$$\frac{v^2(\tau)}{P_w} = \frac{\left| \int_{-\infty}^{\infty} H_R(f)G(f)e^{j2\pi f\tau} df \right|^2}{\int_{-\infty}^{\infty} \eta |H_R(f)|^2 df}$$

- ▶ Schwarz's inequality (A.57) is

$$\left| \int_{-\infty}^{\infty} a(x)b(x)dx \right|^2 \leq \left\{ \int_{-\infty}^{\infty} |a(x)|^2 dx \right\} \left\{ \int_{-\infty}^{\infty} |b(x)|^2 dx \right\}$$

and equality occurs only when $a(x) = kb^*(x)$ where superscript * denotes complex conjugation.

Matched Filter (cont'd)

- ▶ By Schwarz's inequality

$$\frac{v^2(\tau)}{P_w} \leq \frac{\left(\int_{-\infty}^{\infty} |H_R(f)|^2 df \right) \left(\int_{-\infty}^{\infty} |G(f)e^{j2\pi f\tau}|^2 df \right)}{\eta \int_{-\infty}^{\infty} |H_R(f)|^2 df}$$

with the maximum of $v^2(\tau)/P_w$ when

$$H_R(f) = k(G(f)e^{j2\pi f\tau})^*$$

- ▶ Combining the symmetry property (A.35) for real w

$$\mathcal{F}^{-1}\{W^*(-f)\} = w^*(t) \Rightarrow \mathcal{F}^{-1}\{W^*(f)\} = w^*(-t)$$

and the time shift property (A.38)

$$\mathcal{F}^{-1}\{W(f)e^{-j2\pi fT_d}\} = w(t - T_d)$$

yields

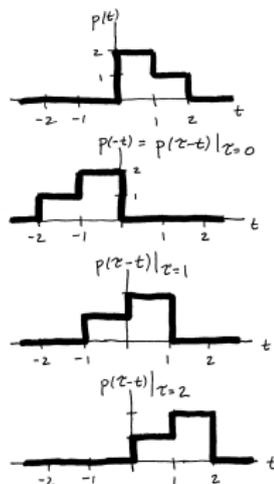
$$\begin{aligned} \mathcal{F}^{-1}\{(W(f)e^{j2\pi fT_d})^*\} &= w^*(-(t - T_d)) \\ &= w^*(T_d - t) \end{aligned}$$

Matched Filter (cont'd)

- ▶ Thus, when $g(t)$ is real

$$\mathcal{F}^{-1}\{k(G(f)e^{j2\pi f\tau})^*\} = kg^*(\tau - t) = kg(\tau - t)$$

- ▶ Example:



Minimum τ for causality of matched filter is pulse width for pulse initiated at $t = 0$.

Matched Nyquist Transmit and Receive Filter Combinations

A preferred receive filter impulse response (in the absence of channel ISI but with broadband channel noise) (i) will match the reversed impulse response of the transmitter pulse shape and (ii) when convolved with the transmitter pulse shape will form a Nyquist pulse.

- ▶ Want convolution of candidate pulse shape $g(t)$ and its matched filter $g(t - \tau)$ to equal even symmetric Nyquist pulse $p(t)$.
- ▶ Since convolution of two even symmetric pulse shapes is even symmetric, presume $g(t)$ is even symmetric, so with particular τ , $g(t) = g(\tau - t)$.
- ▶ Objective becomes

$$p(t) = g(t) * g(t) \Rightarrow P(f) = G^2(f)$$

Matched ... Combinations (cont'd)

- ▶ So, choose

$$G(f) = \sqrt{P(f)} \Rightarrow g(t) = \mathcal{F}^{-1}\{\sqrt{P(f)}\}$$

- ▶ For example, consider the square-root raised cosine (SRRC)

$$v(t) = \begin{cases} \frac{1}{\sqrt{T}} \frac{\sin(\pi(1-\alpha)t/T) + (4\alpha t/T)\cos(\pi(1+\alpha)t/T)}{(\pi t/T)(1-(4\alpha t/T)^2)} & \text{for } t \neq 0, t \neq \pm \frac{T}{4\alpha} \\ \frac{1}{\sqrt{T}}(1 - \alpha + (4\alpha/\pi)) & \text{for } t = 0 \\ \frac{\alpha}{\sqrt{2T}} \left[\left(1 + \frac{2}{\pi}\right) \sin\left(\frac{\pi}{4\alpha}\right) + \left(1 - \frac{2}{\pi}\right) \cos\left(\frac{\pi}{4\alpha}\right) \right] & \text{for } t = \pm \frac{T}{4\alpha} \end{cases}$$

which has a magnitude spectrum the square of which equals the magnitude spectrum of a raised cosine.

- ▶ The square root raised cosine is the most commonly used pulse in bandwidth constrained communication systems.

NEXT... We concoct various timing (aka clock) recovery schemes.