DIGITAL FILTERING AND THE DFT

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Digital Linear Filters in the Receiver

There are a number of places in our PAM communication system receiver after the sampler where a digital linear filter is needed, including:

- lowpass filter in digital downconversion
- pulse-matched filter
- timing interpolator
- equalizer
- correlator for decoder frame synchronization
Discrete-time Linear Systems Tidbits

- **Discrete-time impulse:**
  \[
  \delta[k] = \begin{cases} 
  1 & k = 0 \\
  0 & k \neq 0
  \end{cases}
  \]

- **Signal as weighted sum of delayed impulses:**
  \[
  \{w[k]\} = \{1, 2, -1, \ldots\}
  \]
  \[
  w[k] = \delta[k] + 2\delta[k - 1] - \delta[k - 2] \ldots
  \]

- **Discrete-time linear system response:**
  - input: \( w[k] = \delta[k] \) \( \Rightarrow \) output: \( y[k] = h[k] \)
  - input: \( w[k] = \delta[k] + 2\delta[k - 1] = w[0]\delta[k] + w[1]\delta[k - 1] \)
    \( \Rightarrow \) output: \( y[k] = w[0]h[k] + w[1]h[k - 1] \)
  - input: \( w[k] = \sum_{j=-\infty}^{\infty} w[j] \delta[k - j] \)
    \( \Rightarrow \) output: \( y[k] = \sum_{j=-\infty}^{\infty} w[j] h[k - j] \equiv w[k] \ast h[k] \)
DFT Tidbits

> **DFT:** Given

\[
\{w[k]\} = \{w[0], w[1], \ldots, w[N - 1]\}
\]

we define

\[
W[n] = \text{DFT} (\{w[k]\}) = \sum_{k=0}^{N-1} w[k] e^{-j(2\pi/N)nk} \quad n = 0, 1, 2, \ldots, N - 1
\]

> **IDFT:** Given

\[
\{W[n]\} = \{W[0], W[1], \ldots, W[N - 1]\}
\]

we define

\[
w[k] = \text{IDFT} (\{W[n]\}) = \frac{1}{N} \sum_{n=0}^{N-1} W[n] e^{j(2\pi/N)nk} \quad k = 0, 1, 2, \ldots, N - 1
\]
Define \( w = [w[0] \ w[1] \ w[2] \ ... \ w[N - 1]]^T \)

\[
M^{-1} = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & e^{j2\pi/N} & e^{j4\pi/N} & \ldots & e^{j2\pi(N-1)/N} \\
1 & e^{j4\pi/N} & e^{j8\pi/N} & \ldots & e^{j4\pi(N-1)/N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e^{j2(N-1)\pi/N} & e^{j4(N-1)\pi/N} & \ldots & e^{j2(N-1)^2\pi/N}
\end{bmatrix}
\]

\[
W = [W[0] \ W[1] \ W[2] \ ... \ W[N - 1]]^T
\]

Then, for the IDFT

\[
w = \left(\frac{1}{N}\right)M^{-1}W
\]

and for the DFT

\[
W = NMw
\]
\[ N \mathbf{w} = W[0] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + W[1] \begin{bmatrix} \frac{1}{N} \\ e^{j2\pi/N} \\ \vdots \\ e^{j2(\pi(N-1))/N} \end{bmatrix} + \ldots + W[N - 1] \begin{bmatrix} \frac{1}{N} \\ e^{j2(N-1)\pi/N} \\ \vdots \\ e^{j2(N-1)^2\pi/N} \end{bmatrix} \]

\[ = W[0] \, C_0 + W[1] \, C_1 + \ldots + W[N - 1] \, C_{N - 1} \]

\[ = \sum_{n=0}^{N-1} W[n]C_n \]

\(\mathbf{w}\) is a linear combination of the columns \(C_n\).
DFT Tidbits (cont’d)

- The all ones \( C_0 \) is a vector of samples of a zero frequency sinusoid.
- The entries of \( C_1 \) maintain the same (unit) magnitude but have an angle that increases from 0 to \( 2\pi(N - 1)/N \), i.e. traversing one period over the data record length.
- The entries of \( C_2 \) traverse the full unit circle twice in the positive (counterclockwise) direction.
- The DFT re-expresses the time vector as a linear combination of sinusoids with periods equal to the data record length, half this length, one-third this length, and so forth up to \( (1/(N - 1)) \)th of this length.
DFT Tidbits (cont’d)

The key factors in a DFT based frequency analysis are:

- The sampling interval $T_s$ is the time resolution, the shortest time over which any event can be observed.
- The sampling rate is $f_s = \frac{1}{T_s}$. As the sample rate increases, the time resolution decreases.
- The total time is $T = NT_s$ where $N$ is the number of samples in the analysis.
- The frequency resolution is $\frac{1}{T} = \frac{1}{NT_s} = \frac{f_s}{N}$. Sinusoids closer together (in frequency) than this value are indistinguishable.
- As the time resolution increases, the frequency resolution decreases, and vice versa.
Example: In `specgong.m`

- sampling interval: $T_s = \frac{1}{44100}$
- number of samples: $N = 2^{16}$
- total time: $NT_s = 1.48$ seconds
- frequency resolution: $\frac{1}{NT_s} = 0.67$ Hz
Filter Design Tidbits

Types:
- lowpass filter
- high pass filter
- bandpass filter
- bandstop (notch) filter

*Bandpass filter spectra specification:*
Filter Design Tidbits (cont’d)

From help firpm:
FIRPM performs Parks-McClellan optimal equiripple FIR filter design. i.e. a linear phase (real, symmetric coefficients) FIR filter which has the best approximation to the desired frequency response in the minimax sense. 

\[ b = \text{firpm}(fl,fbe,damps) \]

- \( b \) is the output vector containing the impulse response of the designed filter.
- \( fl \) is (one less than) the number of terms in \( b \).
  - \( fl \uparrow \Rightarrow \) fit to design specs improves
  - \( fl \uparrow \Rightarrow \) computational costs increase
  - \( fl \uparrow \Rightarrow \) throughput delay increases
**Filter Design Tidbits (cont’d)**

- $\text{fbe}$ is a vector of frequency band edge values as a fraction of the prevailing Nyquist frequency. For a basic bandpass filter:
  - bottom of stopband (presumably zero)
  - top edge of lower stopband (which is also the lower edge of the lower transition band)
  - lower edge of passband
  - upper edge of passband
  - lower edge of upper stopband
  - upper edge of upper stopband (generally the last value will be 1).

- $\text{damps}$ is the vector of desired amplitudes of the frequency response at each band edge.
Filter Design Tidbits (cont’d)

From bandex with

- \( \text{fbe} = [0 \ 0.24 \ 0.26 \ 0.74 \ 0.76 \ 1] \)
- \( \text{damps} = [0 \ 0 \ 1 \ 1 \ 0 \ 0] \)
- \( \text{fl} = 30 \)

\( b = \text{firpm}(\text{fl}, \text{fbe}, \text{damps}); \ \text{freqz}(b) \) produces
Filter Design Tidbits (cont’d)

To demonstrate criteria fit impact of filter length, repeat preceding example with $f_l$ halved and doubled.

- $f_l = 15$
Note improved fit to design specifications with increase in filter length ($f_l$).
Filter Design Tidbits (cont’d)

Adding an in-band notch with

- \( fbe=[0 \ 0.24 \ 0.26 \ 0.59 \ 0.595 \ 0.605 \ 0.61 \ 0.74 \ 0.76 \ 1] \)
- \( damps=[0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0] \)
- \( fl=60 \)

Desired notch (at normalized frequency = 0.6) is barely perceptible.
Filter Design Tidbits (cont’d)

- $f_1 = 120$

Note improved fit to design specifications with increase in filter length ($f_1$). Desired notch (at normalized frequency $= 0.6$) is quite pronounced.

NEXT... We discuss the conversion of bits to symbols to pulse-amplitude modulated signals and the reversal with correlation used to locate the frame break.