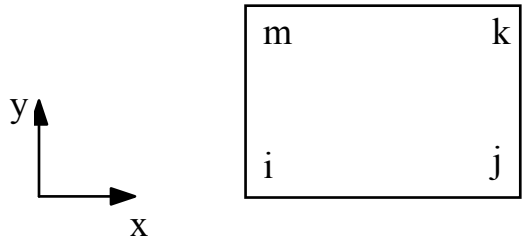
<u>Bilinear Element: Quadrilateral.</u>

4 straight element sides with 4 nodes - 1 at each corner.



Note: m is used instead of lowercase L to eliminate confusion with number 1.

$$N_i = (a_i x + b_i)(c_i y + d_i)$$
$$= \alpha_i + \beta_i x + \gamma_i y + \boldsymbol{\delta_i xy}$$ Note the <u>Bilinear</u> term.

$$\alpha_i = b_i d_i; \quad \beta_i = a_i d_i; \quad \gamma_i = b_i c_i; \quad \delta_i = a_i c_i$$

As with triangles: N_i = 1 at node i

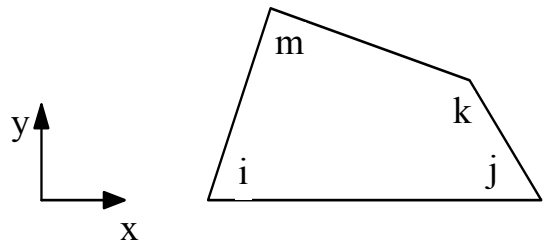
= 0 at nodes j, k, m.

Solve for coefficients:

$$\begin{Bmatrix} 1 & x_i & y_i & xy_i \\ 1 & x_j & y_j & xy_j \\ 1 & x_k & y_k & xy_k \\ 1 & x_m & y_m & xy_m \end{Bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Permutate for i, j, k, m in CCW order.

Consider an irregularly shaped linear quadrilateral:



In general one wants to evaluate:

$$\nabla \cdot (K \nabla U) + \mathbf{V} \cdot \nabla U + f U = g$$

or, numerically

$$\left[< -K(\frac{\partial N_j}{\partial x}\frac{\partial W_i}{\partial x} + \frac{\partial N_j}{\partial y}\frac{\partial W_i}{\partial y}) > + < (V_x \frac{\partial N_j}{\partial x} + V_y \frac{\partial N_j}{\partial y})W_i > + < fN_jW_i > \right] \{U_j\}$$

$$\{< gW_i >\} - \int_S K\nabla U \bullet \mathbf{n} W_i dS$$

However this $\int$ ( )dxdy over each element E will present problems due to the irregular geometry.

SOLUTION: Transform x, y space to <u>local</u> coordinates (ξ, η) and integrate numerically.

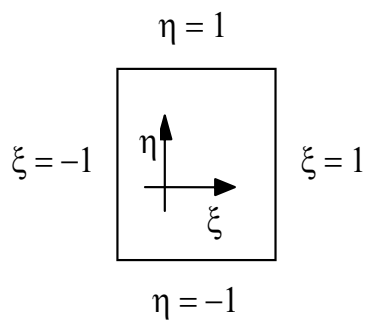
A <u>Bilinear Basis function</u> implies that for any constant x or y the function is linear in the other direction.

let $x = x_c$ = constant

then $N_i = A_i + B_i y$ where $A_i = (\alpha_i + \beta_i x_c)$

$B_i = (\gamma_i + \delta_i x_c)$

Examine a Bilinear local coordinate system:



$\Delta\xi = \Delta\eta = 2$; $\xi = \eta = 0$ at center of element.

$N_i = 1/4\ (1 + \xi_i\xi)(1 + \eta_i\eta)$ which is the bilinear solution of local system.

$$\begin{bmatrix} \text{Coefficient} \\ \text{Matrix} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note: for i = local node 1

$N_1 = 1/4\ (1+(-1)(-1))(1+(-1)(-1)) = 1$ at $\xi = -1, \eta = -1$

$= 1/4\ (1+(-1)(-1))(1+(-1)(1)) = 0$ at $\xi = -1, \eta = +1$

$= 1/4\ (1+(-1)(1))(1+(-1)(1)) = 0$ at $\xi = 1, \eta = 1$

$= 1/4\ (1+(-1)(1))(1+(-1)(-1)) = 0$ at $\xi = 1, \eta = -1$

We need coordinate transformation => use trial function approach.

$$x = \sum_{k=1}^{4} x_k \psi_k(\xi, \eta) \qquad y = \sum_{k=1}^{4} y_k \psi_k(\xi, \eta)$$

with (x, y) being any point within the element and x_k, y_k being the nodal coordinates in (x, y) space. However, ψ_k is formulated in (ξ, η) space.

CONSTRAINTS:

1.) 1 : 1 mapping of nodes between 2 coordinate systems.

2.) linear variation in x and y along sides of (ξ, η) element.

=> $x = a_1 + b_1\xi + c_1\eta + d_1\xi\eta$

$y = a_2 + b_2\xi + c_2\eta + d_2\xi\eta$

Note that solution of ψ_i yields the basis functions N_i

when the transformation (mapping) function = basis function the element is an <u>Isoparametric</u> element

when transformation function is of higher degree than basis function => Superparametric and

when transformation function is of lower degree than basis function => Subparametric.

We need terms such as: $\left[\frac{\partial N}{\partial x} \frac{\partial W}{\partial x} + \ldots\ldots \right]$

i.e express $\frac{\partial(\ )}{\partial x \ or \ \partial y}$ in terms of ξ, η.

Use chain rule:

$$\frac{\partial N(\xi,\eta)}{\partial \xi} = \frac{\partial N(\xi,\eta)}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N(\xi,\eta)}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N(\xi,\eta)}{\partial \eta} = \frac{\partial N(\xi,\eta)}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N(\xi,\eta)}{\partial y}\frac{\partial y}{\partial \eta}$$

or:

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} \qquad [J] \equiv \text{Jacobian transformation matrix.}$$

What we need for evaluation of: $\left[\dfrac{\partial N}{\partial x} \dfrac{\partial W}{\partial x} + \ldots\ldots \right]$, is:

$$\begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix}$$

Recall: $N_i = 1/4\ (1+ \xi_i\xi)(1+ \eta_i\eta)$

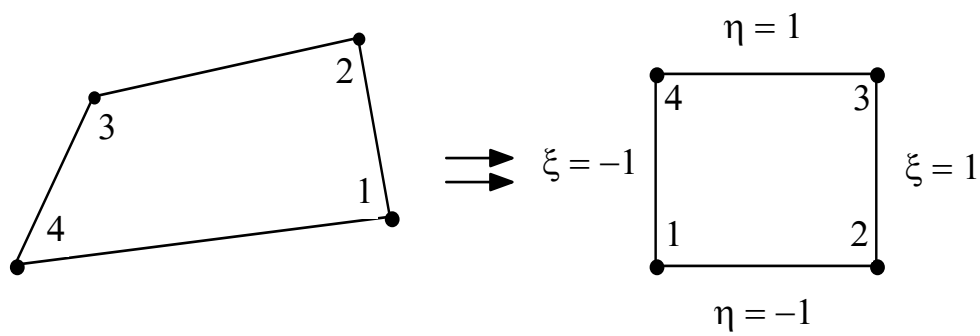
$$\therefore \quad \frac{\partial N_i}{\partial \xi}, \quad \frac{\partial N_i}{\partial \eta} \Rightarrow \quad \text{no problem.}$$

In $[J] \Rightarrow \dfrac{\partial x}{\partial \xi}, etc = ?$

$$x = \sum_{k=1}^{4} x_k \psi_k \qquad \text{and} \qquad \psi_k = N_k \quad \text{(Isoparametric)}$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial \left(\sum x_k \psi_k \right)}{\partial \xi} = \sum_k \frac{\partial (x_k \psi_k)}{\partial \xi} = \sum_k x_k \frac{\partial \psi_k}{\partial \xi} = \sum_k x_k \frac{\partial N_k}{\partial \xi}$$

$$\frac{\partial y}{\partial \xi} = \sum_k y_k \frac{\partial N_k}{\partial \xi}$$



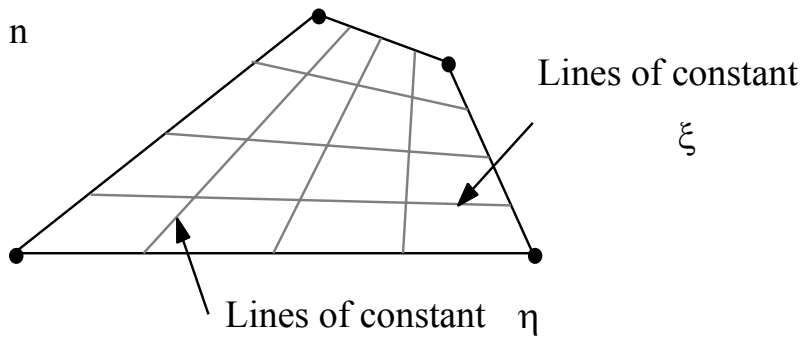
Choose any side; Ex.: $\eta = 1$ = top side in (ξ,η) = left side (x,y)
i.e. node 1 in (x,y) does not need to be lower left.

$$N_i = 1/4\,(1+\xi_i\xi)(1+\eta_i(1))$$
$$= A_i + B_i\xi$$

$$\frac{\partial x}{\partial \xi} = \sum x_i \frac{\partial N_i}{\partial \xi} = \text{Constant}$$

$$\Rightarrow \frac{dx}{dy} = \text{Constant}$$

$$\frac{\partial y}{\partial \xi} = \sum y_i \frac{\partial N_i}{\partial \xi} = \text{Constant}$$



Along any line: η = constant => dx/dy = constant and ξ = constant => same.

$$\begin{Bmatrix} \dfrac{\partial(\ )}{\partial x} \\ \dfrac{\partial(\ )}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial(\ )}{\partial \xi} \\ \dfrac{\partial(\ )}{\partial \eta} \end{Bmatrix} = \frac{1}{|[J]|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & \dfrac{-\partial y}{\partial \xi} \\ \dfrac{-\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial(\ )}{\partial \xi} \\ \dfrac{\partial(\ )}{\partial \eta} \end{Bmatrix}$$

$$\frac{\partial(\ )}{\partial x} = \frac{1}{|J|} \left[\frac{\partial y}{\partial \eta} \frac{\partial(\ )}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial(\ )}{\partial \eta} \right]$$

$$\frac{\partial(\ )}{\partial y} = \frac{1}{|J|} \left[- \frac{\partial x}{\partial \eta} \frac{\partial(\ )}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial(\ )}{\partial \eta} \right]$$

For integration: $\displaystyle\int_a^b \int_c^d (\ )\, dx\, dy = \int_{-1}^{1} \int_{-1}^{1} (\ )\, |J|\, d\xi\, d\eta$

and

$$[J] = \begin{bmatrix} \sum x_i \dfrac{\partial N_i}{\partial \xi} & \sum y_i \dfrac{\partial N_i}{\partial \xi} \\ \sum x_i \dfrac{\partial N_i}{\partial \eta} & \sum y_i \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$