Stability: Fourier Method (Von Neumann)

Let U represent the solution to the PDE (or any finite difference approximation). Any function (U analytic, U numeric, or any form of error, be it a small perturbation introduced or the difference between $U_{numeric}$-$U_{analytic}$) can be represented by a Fourier series. If the function is finite (i.e. point to point), then the Fourier series is finite too.

- Assume a separable solution $U = Ae^{\alpha t}e^{j\sigma\chi}$ $j = \sqrt{-1}$
- Plug into PDE will result in a dispersion relation between α and σ
- Synthesize Solution (Linearity, Superposition)

$$U(\chi, t) = \sum_i A_i e^{\alpha_i t} e^{j\sigma_i x}$$

Since we are only interested in finding any harmonic that goes unstable, the index (i) can be dropped from the formulation.

Recall: $\sigma = \frac{2\pi}{L}$ $2h \le L < \infty$

$$\Rightarrow \quad 0 < \sigma \le \frac{\pi}{h}$$

Distributed System (PDE)

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \quad \Rightarrow \quad \boxed{\alpha = -D\sigma^2}$$

- Stable system ($\alpha < 0$ time amplifier)

  $\therefore$ solution smooths over time
- Longest waves (L or $f(1/\sigma)$) decay slowest

Lumped Systems: Spatial Discretizations, O.D.E.

$$\frac{dU_i}{dt} = \frac{D}{h^2}\,\delta_x^2 U_i = \frac{D}{h^2}\,(U_{i-1} - 2U_i + U_{i+1})$$

Recall: $U = Ae^{\alpha t}e^{j\sigma\chi} \quad \Rightarrow \quad \delta_x^2 U_i = (e^{-j\sigma h} - 2 + e^{j\sigma h})\,U_i$

$$= 2(\cos(\sigma h) - 1)\,U_i$$

Then

$$\alpha U_i = \frac{2D}{h^2}\,(\cos(\sigma h) - 1)\,U_i$$

or

$$\alpha = -D\sigma^2 \underbrace{\left[\frac{2(1 - \cos(\sigma h))}{\sigma^2 h^2}\right]}_{\text{Effect of Lumping (FD in }\chi)}$$

Use series expansions for transcendental functions (like exp, trig., etc.) to see where the finite difference approximation deviates from the PDE.

$$\mathrm{Cos}(\sigma h) = 1 - \frac{(\sigma h)^2}{2!} + \frac{(\sigma h)^4}{4!} - \cdots\cdots$$

Retain $O(h^2)$ for any σ $\longrightarrow$ h.o.t

$$\alpha = -D\sigma^2\left[1 - \frac{2(\sigma h)^2}{4!} + \cdots\right] = -D\sigma^2\left[1 - \frac{\left(\frac{2\pi h}{L}\right)^2}{12} + \cdots\right]$$

Introduce the Propagation Factor, γ_θ

Expressing U as: $U = Ae^{\alpha t}e^{j\sigma\chi}$ $\quad j = \sqrt{-1}$

Distributed System: $\alpha = -D\sigma^2$

Lumped System $\quad \alpha' = -D\sigma^2\left[1 - \frac{(\sigma h)^2}{12} + \cdots\right]$

$$r = \frac{D\Delta t}{h^2} \qquad \frac{U(t+\Delta t)}{U(t)} = e^{\alpha\Delta t} \equiv \gamma$$

Distributed: $\gamma = e^{-D\sigma^2\Delta t} = e^{-r(\sigma h)^2}$

Lumped: $\gamma' = e^{-D\sigma^2\Delta t\left[1 - \frac{(\sigma h)^2}{12} + \cdots\right]} = \gamma e^{+D\sigma^2\Delta t\frac{(\sigma h)^2}{12}} \cdot e^{(\,)}$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\gamma' = \gamma\left[1 + D\sigma^2\Delta t\,\frac{(\sigma h)^2}{12} + \cdots\right]\cdot[1 + \cdots]\cdot[\;]\cdots$$

- error ~ Δt
- error(t) ~ h^2

Discrete System: e.g. as a function of θ

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \qquad \frac{dU_i}{dt} = \frac{D}{h^2}\delta_x^2 U_i \qquad \frac{U_i^{k+1} - U_i^k}{\Delta t} = \frac{D}{h^2}\delta_x^2 U_i^{k+\theta}$$

Distributed Lumped Discrete

Where $U_i^{k+\theta} = \theta U_i^{k+1} + (1-\theta)U_i^k$

$$U_i^{k+1} - U_i^k = r\left[\theta(U_{i+1} - 2U_i + U_{i-1})^{k+1} + (1-\theta)(U_{i+1} - 2U_i + U_{i-1})^k\right]$$

$$U_i^{k+1} - r\theta(U_{i+1} - 2U_i + U_{i-1})^{k+1} = U_i^k + r(1-\theta)(U_{i+1} - 2U_i + U_{i-1})^k$$

$$U_i^{k+1} - 2r\theta(Cos(\sigma h) - 1)U_i^{k+1} = U_i^k + 2r(1-\theta)(Cos(\sigma h) - 1)U_i^k$$

Using $\frac{\mathrm{U(t + \Delta t)}}{\mathrm{U(t)}} = \mathrm{e}^{\alpha \Delta \mathrm{t}} \equiv \gamma$

$$\gamma_\theta[1 - 2r\theta(Cos(\sigma h) - 1)] = 1 + 2r(1-\theta)(Cos(\sigma h) - 1)$$

$$\gamma_\theta = \frac{1 - 2r(1-\theta)(1 - Cos(\sigma h))}{[1 + 2r\theta(1 - Cos(\sigma h))]}$$

For stability $|\gamma_\theta| < 1$ $\qquad\gamma_\theta < 1 \quad Always$

Therefore, the question is:

$$\gamma_\theta = \frac{1 - 2r(1-\theta)(1 - Cos(\sigma h))}{[1 + 2r\theta(1 - Cos(\sigma h))]} > -1 \qquad for \quad stability$$

$$1 - 2r(1-\theta)(1 - Cos(\sigma h)) > -1 - 2r\theta(1 - Cos(\sigma h))$$

$$-r(1-\theta)(1 - Cos(\sigma h)) > -1 - r\theta(1 - Cos(\sigma h))$$

$$-r(1-2\theta)(1 - Cos(\sigma h)) > -1 \quad \text{or} \quad r(1-2\theta) < \frac{1}{(1 - Cos(\sigma h))}$$

For θ < ½ the quantity (1-2θ) is positive

$$r < \frac{1}{(1-2\theta)(1-Cos(\sigma h))}$$

$$r < \frac{1}{2(1-2\theta)}$$

Conditional Stability

For θ ≥ ½ the quantity (1-2θ) is negative

$$r > \frac{1}{(1-2\theta)(1-Cos(\sigma h))}$$

$$r > \frac{1}{2(1-2\theta)}$$

Unconditional Stability
r > (negative number)

What happens in multiple spatial dimensions?

$$U = A\exp(\alpha t)\exp(j\sigma x)\exp(j\beta y)$$

$$\sigma = \frac{2\pi}{L_x} \qquad 2h < L_x \ \ and/or \ \ L_y < \infty \qquad \beta = \frac{2\pi}{L_y}$$

$$\begin{aligned} U_{i,j}^{k+1} - U_{i,j}^{k} &= r\theta(U_{i+1,j} - 2U_{i,j} + U_{i-1,j})^{k+1} + r\theta(U_{i,j+1} - 2U_{i,j} + U_{i,j-1})^{k+1} \\ &\quad + r(1-\theta)(U_{i+1,j} - 2U_{i,j} + U_{i-1,j})^{k} + r(1-\theta)(U_{i,j+1} - 2U_{i,j} + U_{i,j-1})^{k} \end{aligned}$$

$$\begin{aligned} U_{i,j}^{k+1} - U_{i,j}^{k} &= 2r\theta(\cos(\sigma h) - 1)U_{i,j}^{k+1} + 2r\theta(\cos(\beta h) - 1)U_{i,j}^{k+1} \\ &\quad + 2r(1-\theta)(\cos(\sigma h) - 1)U_{i,j}^{k} + 2r(1-\theta)(\cos(\beta h) - 1)U_{i,j}^{k} \end{aligned}$$

$$U_{i,j}^{k+1} - U_{i,j}^{k} = 2r\theta(\cos(\sigma h)-1)U_{i,j}^{k+1} + 2r\theta(\cos(\beta h)-1)U_{i,j}^{k+1}$$
$$+2r(1-\theta)(\cos(\sigma h)-1)U_{i,j}^{k} + 2r(1-\theta)(\cos(\beta h)-1)U_{i,j}^{k}$$

Note that: $\dfrac{U_{i,j}^{k+1}}{U_{i,j}^{k}} = \exp(\alpha\Delta t) \equiv \gamma$ has not changed.

$$\gamma_\theta - 1 = 2r\theta\left[(\cos(\sigma h)-1) + (\cos(\beta h)-1)\right]\gamma_\theta$$
$$+2r(1-\theta)\left[(\cos(\sigma h)-1) + (\cos(\beta h)-1)\right]$$

$$\gamma_\theta\left[1 - 2r\theta\left[(\cos(\sigma h)-1) + (\cos(\beta h)-1)\right]\right] =$$
$$1 + 2r(1-\theta)\left[(\cos(\sigma h)-1) + (\cos(\beta h)-1)\right]$$

or

$$\gamma_\theta\left[1 + 2r\theta\left[(1-\cos(\sigma h)) + (1-\cos(\beta h))\right]\right] =$$
$$1 - 2r(1-\theta)\left[(1-\cos(\sigma h)) + (1-\cos(\beta h))\right]$$

$$0 \le C = \left[(1-\cos(\sigma h)) + (1-\cos(\beta h))\right] \le 4 \qquad non-negative$$

$$\gamma_\theta\left[1 + 2r\theta C\right] = 1 - 2r(1-\theta)C$$

$$\left|\gamma_\theta = \frac{1-2r(1-\theta)C}{\left[1+2r\theta C\right]}\right| \le 1 \qquad for\ Stability$$

$$\gamma_\theta \le 1 \quad always$$

$$\gamma_\theta = \frac{1-2r(1-\theta)C}{\left[1+2r\theta C\right]} \ge -1 \qquad is\ the\ question$$

$$1-2r(1-\theta)C \ge -1-2r\theta C$$

$$-2r(1-2\theta)C \ge -2$$

$$or \qquad r(1-2\theta) \le \frac{1}{C}$$

Stability constraints will occur most when C=4

For θ < ½ the quantity
(1-2θ) is positive

$$r < \frac{1}{4(1-2\theta)}$$

$$r_{2D} = \frac{r_{1D}}{2}$$

More severe
conditional stability

For θ ≥ ½ the quantity
(1-2θ) is negative

$$r > \frac{1}{4(1-2\theta)}$$

Unconditional Stability
r > (negative number)

Consider a three-level time stepping scheme:

$$\frac{U_i^{k+1}-U_i^{k-1}}{2\Delta t} = \frac{D}{h^2}\delta_x^2 U_i^{k+\theta} = -2\frac{D}{h^2}(1-\cos(\sigma h))U_i^{k+\theta}$$

$$U_i^{k+\theta} = \frac{\theta}{2}U_i^{k+1} + (1-\theta)U_i^{k} + \frac{\theta}{2}U_i^{k-1}$$

$$\gamma_\theta^2 - 1 = -4r(1-\cos(\sigma h))\left[\frac{\theta}{2}\gamma_\theta^2 + (1-\theta)\gamma_\theta + \frac{\theta}{2}\right]$$

$$
\begin{aligned}
&(1+4r(1-\cos(\sigma h)\frac{\theta}{2})\gamma_\theta^2 \\
&\qquad +4r(1-\cos(\sigma h))(1-\theta)\gamma_\theta \\
&\qquad\qquad +(-1+4r(1-\cos(\sigma h)\frac{\theta}{2})=0
\end{aligned}
$$

or $$a\gamma_\theta^2+b\gamma_\theta+c=0$$