Last time: Parabolic PDEs were introduced.
Solution of Parabolic PDEs propagate from ICs and are constrained by BCs.

Various time weightings were introduced with the most general formulation as:

\[-r\theta, 1 + 2r\theta, -r\theta, -r(1 - \theta), -1 + 2r(1 - \theta), -r(1 - \theta)\]

with \(\theta = 0.5\)

Crank Nicolson 0(h^2 + k^2)

3 Classic Definitions

Convergence \(U_i^m \rightarrow \mu(x_i, t^m)\)

Consistency \(L_i \rightarrow L\)

Stability \(U_i\) bounded for bounded B. C.

Discrete System Convergence on an Euler Explicit Scheme

\[\|\varepsilon\|^m \leq A(k + h^2) mk = A(k + h^2) t\] at given point in time
Stability: Fourier Method  (Von Neumann)
as with elliptic systems

Let \( U \) represent the solution to the PDE (or any finite difference approximation). Any function (\( U \) analytic, \( U \) numeric, or any form of error, be it a small perturbation introduced or the difference between \( U_{\text{numeric}} - U_{\text{analytic}} \)) can be represented by a Fourier series. If the function is finite (i.e. point to point), then the Fourier series is finite too.

• Assume a separable solution  \( U = A e^{\alpha t} e^{j\sigma \chi} \quad j = \sqrt{-1} \)
• Plug into PDE will result in a dispersion relation between \( \alpha \) and \( \sigma \)
• Synthesize Solution (Linearity, Superposition)
  \[
  U(\chi, t) = \sum_i A_i e^{\alpha_i t} e^{j\sigma_i \chi}
  \]

Since we are only interested in finding any harmonic that goes unstable, the index \( (i) \) can be dropped from the formulation.

Recall:  \( \sigma = \frac{2\pi}{L} \quad 2h \leq L < \infty \)
\[
\Rightarrow \quad 0 < \sigma \leq \frac{\pi}{h}
\]

Distributed System (PDE)
\[
\frac{\partial U}{\partial t} = D \frac{\partial^2 U}{\partial x^2} \quad \Rightarrow \quad \alpha = -D\sigma^2
\]
- Stable system \((\alpha < 0 \text{ time amplifier})\)
  \( \therefore \) solution smooths over time
- Longest waves (L or f(1/σ)) decay slowest
  for large L : \( \sigma \rightarrow 0 \rightarrow |\alpha| \) becomes small
  \( e^{\alpha t} \) decay is slow

Lumped Systems: Spatial Discretizations, O.D.E.

\[
\frac{dU_i}{dt} = \frac{D}{h^2} \delta_x^2 U_i = \frac{D}{h^2} (U_{i-1} - 2U_i + U_{i+1})
\]

Recall: \( U = Ae^{\alpha t}e^{j\sigma \chi} \) \( \Rightarrow \delta_x^2 U_i = (e^{-j\sigma h} - 2 + e^{j\sigma h}) U_i = 2(\cos(\sigma h) - 1) U_i \)

Then

\[
\alpha U_i = \frac{2D}{h^2} (\cos(\sigma h) - 1) U_i
\]

or

\[
\alpha = -D\sigma^2 \left[ \frac{2(1 - \cos(\sigma h))}{\sigma^2 h^2} \right]
\]

Effect of Lumping (FD in \( \chi \))

Use series expansions for transcendental functions (like exp, trig., etc.) to see where the finite difference approximation deviates from the PDE.

\[
\cos(\sigma h) = 1 - \frac{(\sigma h)^2}{2!} + \frac{(\sigma h)^4}{4!} - \cdots
\]

Retain \( O(h^2) \) for any \( \sigma \rightarrow h.o.t \)
\[
\alpha = -D\sigma^2 \left[ 1 - \frac{2(\sigma h)^2}{4!} + \cdots \right] = -D\sigma^2 \left[ 1 - \frac{(2\pi h)^2}{12} + \cdots \right]
\]

Accuracy depends on \( \sigma h = \frac{2\pi h}{L} \)

i.e. \( h \) meaningful only relative to \( L \)

Also since \( 2h \leq L \) all \( \alpha < 0 \) \( \Rightarrow \) stable

Compare Distributed \( \left( \frac{\alpha}{-D\sigma^2} \right) \) and Lumped Systems

a.) Lumped is under damped relative to Distributed
b.) Error is greatest as small \( L/h \)
Introduce the Propagation Factor, $\gamma_0$

Expressing $U$ as:  
$$U = Ae^{\alpha t} e^{j\sigma \chi} \quad j = \sqrt{-1}$$

Distributed System:  
$$\alpha = -D\sigma^2$$

Lumped System  
$$\alpha' = -D\sigma^2 \left[ 1 - \frac{(\sigma h)^2}{12} + \cdots \right]$$

$$r = \frac{D\Delta t}{h^2} \quad \frac{U(t + \Delta t)}{U(t)} = e^{\alpha \Delta t} \equiv \gamma$$

Distributed:  
$$\gamma = e^{-D\sigma^2 \Delta t} = e^{-r(\sigma h)^2}$$

Lumped:  
$$\gamma' = e^{-D\sigma^2 \Delta t \left[ 1 - \frac{(\sigma h)^2}{12} + \cdots \right]} = \gamma e^{+D\sigma^2 \Delta t \frac{(\sigma h)^2}{12}} \cdot e(\cdot)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\gamma' = \gamma \left[ 1 + D\sigma^2 \Delta t \frac{(\sigma h)^2}{12} + \cdots \right] \cdot [1 + \cdots] \cdot [ ] \cdots$$

- error $\sim t$
- error$(t) \sim h^2$
Discrete System: e.g. Euler explicit

\[ U_{i}^{k+1} = U_{i}^{k} + r \delta_{x}^{2} U_{i}^{k} \]

\[ U_{i}^{k+1} = \gamma_{o} U_{i}^{k} \quad ; \quad \delta_{x}^{2} U_{i} = 2(\cos \sigma h - 1) U_{i} \]

\[ \Rightarrow \quad \gamma_{o} U_{i}^{k} = U_{i}^{k} - 2r(1 - \cos \sigma h) U_{i}^{k} \]

\[ \gamma_{o} = 1 - 2r(1 - \cos \sigma h) \quad 0 \leq \sigma h \leq \pi \]

\[ \gamma_{o} \leq 1 \quad \text{Always} \]

\( \gamma_{o} \) negative when \( 1 - 2r(1 - \cos \sigma h) < 0 \)

if \( \gamma_{o} \) is <0 (but > -1) this will result in oscillations in time. For the Euler explicit formulation, these oscillations will occur when:

\[ \frac{1}{2(1 - \cos \sigma h)} < r \]

i.e. when \( r > \frac{1}{4} \) \( \Rightarrow \) the shortest waves will begin to oscillate.

If \( \gamma_{o} \) is <-1 then the error is increasing every time step and the formulation will be unstable. For the Euler explicit formulation, this instability will occur when:

\[ |\gamma_{o}| > 1 \quad \text{when} \quad \gamma_{o} < -1 \]

This situation is entirely a numerical artifact; it has no basics in reality (PDE).
\[ 1 - 2r(1 - \cos\sigma h) < -1 \]
\[ 2 < 2r(1 - \cos\sigma h) \]

\[ \frac{1}{(1 - \cos\sigma h)} < r \]

That is, when \( r > \frac{1}{2} \) \( \Rightarrow \) the shortest waves have unstable oscillations.

Stability criterion

\[ r < \frac{1}{2} \]

Note: dimensionless form

Recall: Discrete system convergence

\[ |\varepsilon_i|^{k+1} \leq |r| \cdot |\varepsilon_{i-1}|^k \]
\[ + |1 - 2r| \cdot |\varepsilon_i|^k \]
\[ + |r| \cdot |\varepsilon_{i+1}|^k \]

\[ r < \frac{1}{2} \]

\[ |1 - 2r| = 1 - 2r \]