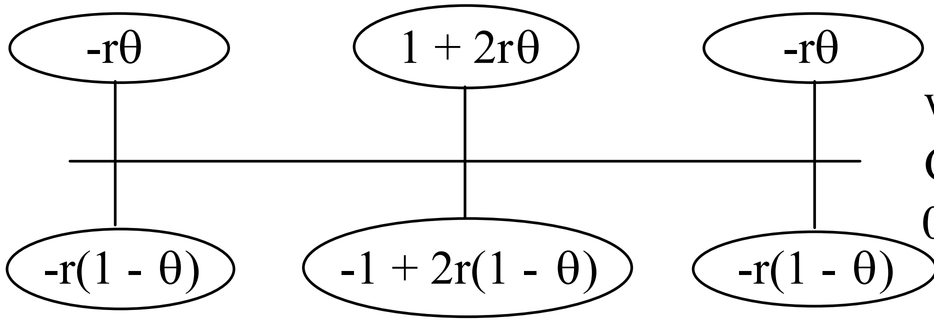
Last time : Parabolic PDEs were introduced
Solution of Parabolic PDEs propagate from ICs and are constrained by BCs

Various time weightings were introduced with the most general formulation as:



with $\theta = 0.5$
Crank Nicolson
$0(h^2 + k^2)$

3 Classic Definitions

Convergence $U_i^m \rightarrow \mu(x_i, t^m)$

Consistency $L_i \rightarrow L$

Stability U_i bounded for bounded B. C.

Discrete System Convergence on an Euler Explicit Scheme

$\| \varepsilon \|^m \leq A(k + h^2)\, mk = A(k + h^2)\, t$ at given point in time

Stability: Fourier Method (Von Neumann)

as with elliptic systems

Let U represent the solution to the PDE (or any finite difference approximation). Any function (U analytic, U numeric, or any form of error, be it a small perturbation introduced or the difference between $U_{numeric}$-$U_{analytic}$) can be represented by a Fourier series. If the function is finite (i.e. point to point), then the Fourier series is finite too.

- Assume a separable solution $U = Ae^{\alpha t}e^{j\sigma\chi}$ $j = \sqrt{-1}$
- Plug into PDE will result in a dispersion relation between α and σ
- Synthesize Solution (Linearity, Superposition)

$$U(\chi, t) = \sum_i A_i e^{\alpha_i t} e^{j\sigma_i x}$$

Since we are only interested in finding any harmonic that goes unstable, the index (i) can be dropped from the formulation.

Recall: $\sigma = \dfrac{2\pi}{L}$ $2h \le L < \infty$

$$\Rightarrow \quad 0 < \sigma \le \frac{\pi}{h}$$

Distributed System (PDE)

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \quad \Rightarrow \quad \boxed{\alpha = -D\sigma^2}$$

- Stable system ($\alpha < 0$ time amplifier)

$\therefore$ solution smooths over time

- Longest waves (L or $f(1/\sigma)$) decay slowest
  - for large L : $\sigma \to 0 \to |\alpha|$ becomes small
  - $e^{\alpha t}$ decay is slow

Lumped Systems: Spatial Discretizations, O.D.E.

$$\frac{dU_i}{dt} = \frac{D}{h^2}\delta_x^2 U_i = \frac{D}{h^2}(U_{i-1} - 2U_i + U_{i+1})$$

Recall: $U = Ae^{\alpha t}e^{j\sigma\chi} \Rightarrow \delta_x^2 U_i = (e^{-j\sigma h} - 2 + e^{j\sigma h})\, U_i$
$= 2(\cos(\sigma h) - 1)\, U_i$

Then

$$\alpha U_i = \frac{2D}{h^2}(\cos(\sigma h) - 1)\, U_i$$

or

$$\alpha = -D\sigma^2 \underbrace{\left[\frac{2(1 - \cos(\sigma h))}{\sigma^2 h^2}\right]}_{\text{Effect of Lumping (FD in } \chi)}$$

Use series expansions for transcendental functions (like exp, trig., etc.) to see where the finite difference approximation deviates from the PDE.

$$\text{Cos}(\sigma h) = 1 - \frac{(\sigma h)^2}{2!} + \frac{(\sigma h)^4}{4!} - \cdots\cdots$$

Retain $O(h^2)$ for any σ → h.o.t

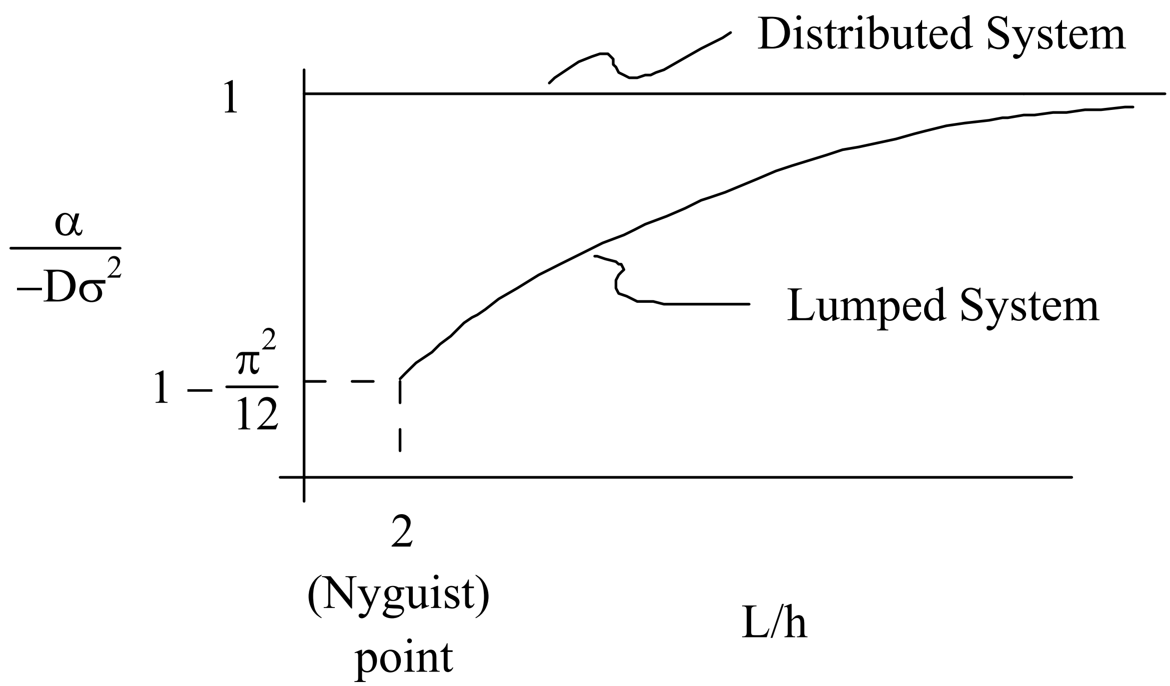
$$\alpha = -D\sigma^2\left[1 - \frac{2(\sigma h)^2}{4!} + \cdots\right] = -D\sigma^2\left[1 - \frac{\left(\frac{2\pi h}{L}\right)^2}{12} + \cdots\right]$$

Accuracy depends on $\sigma h = \frac{2\pi h}{L}$

i.e. h meaningful only relative to L

Also since $2h \leq L$ all $\alpha < 0$ $\Rightarrow$ stable

Compare Distributed $\left(\frac{\alpha}{-D\sigma^2}\right)$ and Lumped Systems



a.) Lumped is under damped relative to Distributed

b.) Error is greatest as small L/h

Introduce the Propagation Factor, γ_θ

Expressing U as: $U = Ae^{\alpha t}e^{j\sigma\chi}$ $\quad j = \sqrt{-1}$

Distributed System: $\alpha = -D\sigma^2$

Lumped System $\alpha' = -D\sigma^2\left[1 - \frac{(\sigma h)^2}{12} + \cdots\right]$

$$r = \frac{D\Delta t}{h^2} \qquad \frac{U(t + \Delta t)}{U(t)} = e^{\alpha\Delta t} \equiv \gamma$$

Distributed: $\gamma = e^{-D\sigma^2\Delta t} = e^{-r(\sigma h)^2}$

Lumped: $\gamma' = e^{-D\sigma^2\Delta t\left[1 - \frac{(\sigma h)^2}{12} + \cdots\right]} = \gamma e^{+D\sigma^2\Delta t\frac{(\sigma h)^2}{12}} \cdot e^{(\;)}$

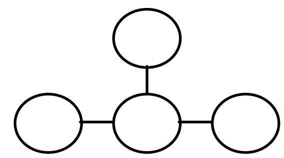
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\gamma' = \gamma\left[1 + D\sigma^2\Delta t\,\frac{(\sigma h)^2}{12} + \cdots\right]\cdot[1 + \cdots]\cdot[\;]\cdots$$

- error ~ t
- error(t) ~ h^2

Discrete System: e.g. Euler explicit

$$U_i^{k+1} = U_i^k + r\delta_x^2 U_i^k$$



$$U_i^{k+1} = \gamma_o U_i^k \quad ; \quad \delta_x^2 U_i = 2(\cos\sigma h - 1)\, U_i$$

$$\Rightarrow \quad \gamma_o U_i^k = U_i^k - 2r(1 - \cos\sigma h)\, U_i^k$$

$$\gamma_o = 1 - 2r(1 - \cos\sigma h) \qquad 0 \le \sigma h \le \pi$$

$\gamma_o \le 1$ Always

γ_o negative when $1 - 2r(1 - \cos\sigma h) < 0$

if γ_o is <0 (but > -1) this will result in oscillations in time. For the Euler explicit formulation, these oscillations will occur when:

$$\boxed{\frac{1}{2(1 - \cos\sigma h)} < r}$$

i.e. when $r > \frac{1}{4} \Rightarrow$ the shortest waves will begin to oscillate.

If γ_o is < -1 then the error is increasing every time step and the formulation will be unstable. For the Euler explicit formulation, this instability will occur when:

$$|\gamma_o| > 1 \text{ when } \gamma_o < -1$$

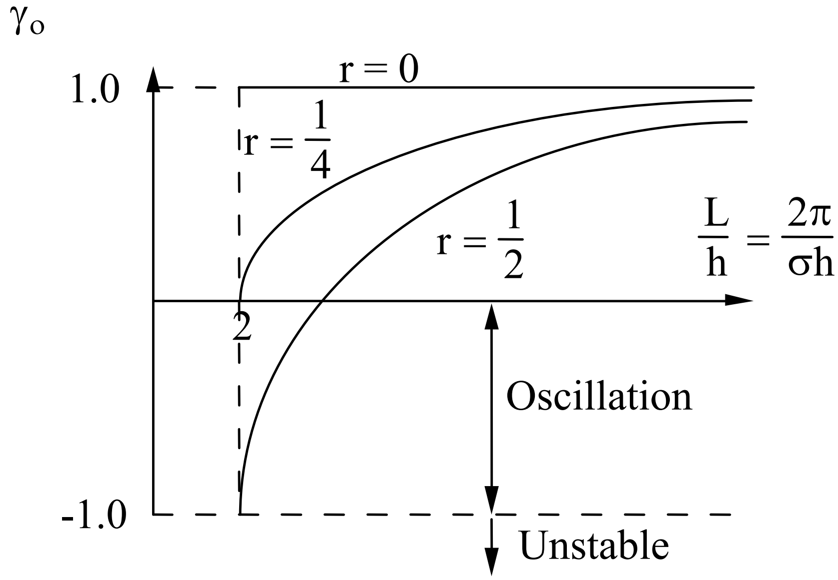
This situation is entirely a numerical artifact; it has no basics in reality (PDE).

$$1 - 2r(1 - \cos\sigma h) < -1$$
$$2 < 2r(1 - \cos\sigma h)$$

$$\frac{1}{(1 - \cos\sigma h)} < r$$

That is, when $r > \frac{1}{2} \Rightarrow$ the shortest waves have <u>unstable</u> oscillations

Stability criterion $r < \frac{1}{2}$



Note: dimentionless form

Recall: Discrete system convergence

$$|\varepsilon_i|^{k+1} \le |r|\,|\varepsilon_{i-1}|^k + |1 - 2r|\,|\varepsilon_i|^k + |r|\,|\varepsilon_{i+1}|^k$$

$$r < \frac{1}{2}$$

$$|1 - 2r| = 1 - 2r$$