Parabolic P.D.E

Parabolic P.D.E arising in scientific and engineering problems are often of the form

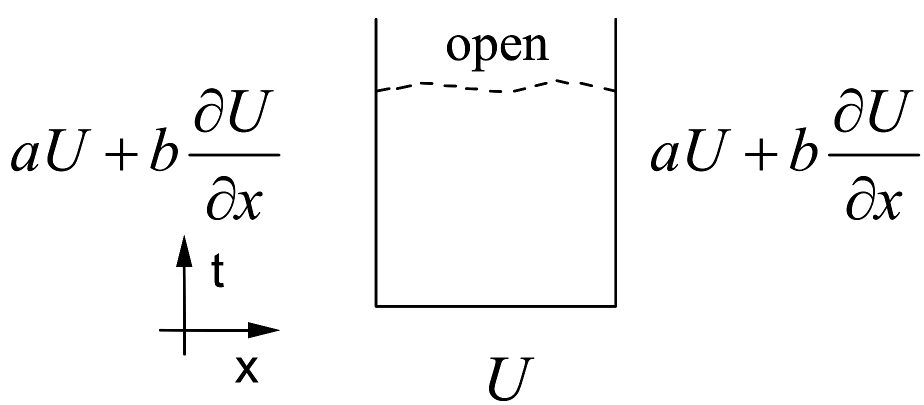
$$U_t = L(U)$$

where L(U) is a 2^{nd} order elliptic partial differential operator which may be linear or nonlinear.

Applications

Diffusions in an isotropic medium; heat conduction in an isotropic medium; fluid flow through porous media; boundary layer flow over a flat plate; persistence of solar prominences wake growth behind a submerged object.

$$U_t = \nabla\cdot D\nabla U$$



The solution of a Parabolic PDE propagates from the Initial Conditions and is constrained by the Boundary Conditions

Continuum ; P.D.E. ; Distributed System

$$\frac{\partial U}{\partial t} = L(U)$$

Numerically:

1.) discretize spatial domain creating a Lumped System

$U(x,t) \rightarrow U_i(t)$

$L(U) \rightarrow L_i(U_i)$

This stage is useful for applying B.C. and conservation

2.) time: create discrete system

a.) F.D. in time

$$\frac{U_i(t+\Delta t) - U_i(t)}{\Delta t} = L_i(U_i)^*$$

← Evaluated at some point on interval $t \rightarrow t + \Delta t$

There are alternate views of same thing

b.) $\int dt$

$$U_i(t+\Delta t) - U_i(t) = \Delta t \, \overline{L_i(U_i)}$$

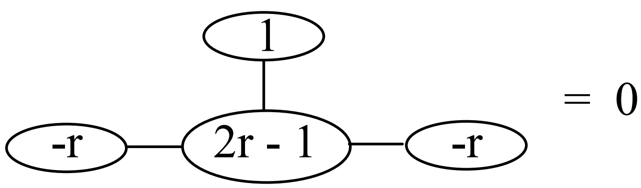
Ex. The k in $\delta_x^2 U_i^k$ implies explicit

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \qquad \frac{\partial U}{\partial t} = \frac{D}{h^2}\delta_x^2 U_i \qquad U_i^{k+1} - U_i^k = \frac{D\Delta t}{h^2}\delta_x^2 U_i^k$$

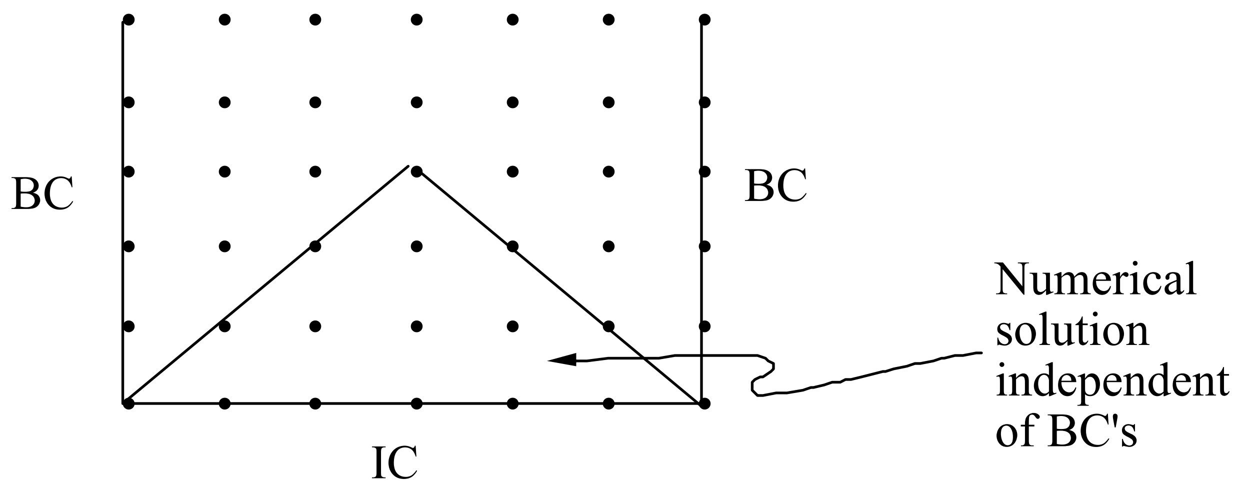
Distributed Lumped Discrete

Molecule for this example: let $\frac{D\Delta t}{h^2} \equiv r$

$$U_i^{k+1} - U_i^k = rU_{i-1}^k - 2rU_i^k + rU_{i+1}^k$$



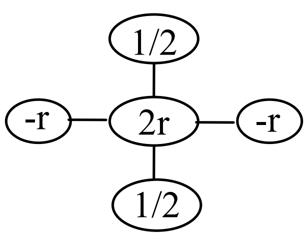
- "Explicit"
- Pointwise propagate
- $0(k + h^2)$
- Forward Diff. in t
- Euler $\int dt$
- Conditional Stability



Expect a critical value of $k \left(= \Delta t\right)$ which must not be exceeded

<u>All</u> pointwise propagation schemes (explicit schemes) have this property: solution runs ahead of itself. (Appropriate for Hyperbolic)

FD in t $\int dt$

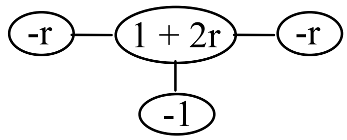


"Leapfrog" ; "Richardson"   Centered Difference   Midpoint

$$\frac{U_i^{k+1} - U_i^{k-1}}{2} = r\,\delta_x^2 U_i^k$$

$0\ h^2 + k^2$

Unconditionally Unstable ! ← Perfect ex. of intuition gone astray



"Implicit"

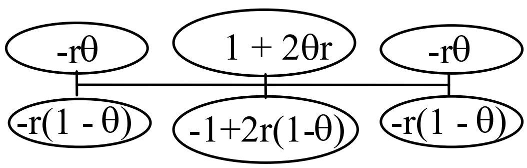
Backward Difference

Euler Backward

$U_i^{k+1} - U_i^k = r\delta_x^2 U_i^{k+1}$

Unconditionally
Stable

for $(\theta = .5)$



"Crank-Nicolson" Centered

Trapezoidal Rule

0 $h^2 + k^2$

$U_i^{k+1} - U_i^k = r\theta\delta_x^2 U_i^{k+1} + r\ 1 - \theta\ \delta_x^2 U_i^k$

$\theta \geq .5$ Unconditional

Stability

$\theta < .5$ Conditional

BC's - as in elliptic problem.

ex. at i = 0 : $-D\dfrac{\partial U}{\partial x} = q_0$ Type II or III

$$\frac{U_{-1} - U_1}{2h} = \frac{q_0}{D}$$

Plus PDE template : $\dfrac{\partial U}{\partial t} = D\dfrac{\partial^2 U}{\partial x^2}$

$$\frac{dU_0}{dt} = \frac{D}{h^2}\left(U_{-1} - 2U_0 + U_1\right)$$

$$U_{-1} = \frac{2hq_0}{D} + U_1$$

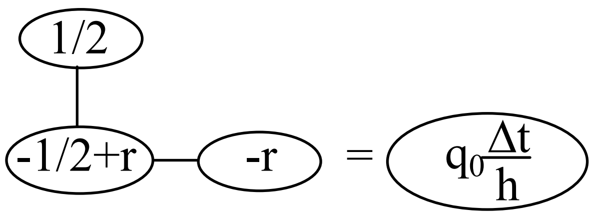
Combine :

$$\frac{dU_0}{dt} = \frac{D}{h^2}\left(2U_1 - 2U_0 + \frac{2hq_0}{D}\right)$$

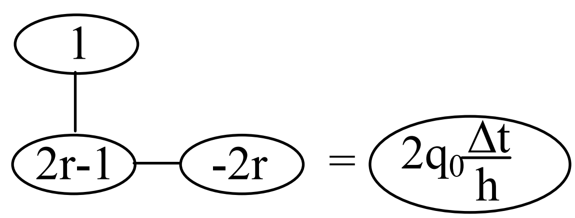
$$\frac{h}{2}\frac{dU_0}{dt} = D\,\frac{\left(U_1 - U_0\right)}{h} + q_0$$

(Compare with previous lecture $\sigma \sim -\dfrac{dU}{dt}$)

Explicit (Euler) Molecule at node 0 :



or



Note q_0 evaluated at level l here.

3 Classic Definitions

a.) <u>Convergence</u> : $U_i^k \to U(x_i, t^k)$ as $h, k \to 0$ independently

b.) <u>Consistency</u> : $L_i \to L$ as $h, k \to 0$ independently
(includes time part, too)
Weaker than Convergence ; easier to show
Essentially : FD molecule $\to$ PDE

c.) <u>Stability</u> : U_i^n bounded for bounded BC's, IC's, forcing.

For linear problems : Homogeneous response
to IC's $\to 0$ at large t.

Rule of thumb : (b) + (c) $\to$ (a)

Discrete System Convergence on an Euler Explicit Scheme

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \qquad \frac{dU}{dt} = \frac{D}{h^2}\delta_x^2 U_i \qquad U_i^{k+1} - U_i^k = \frac{D\Delta t}{h^2}\delta_x^2 U_i^k$$

Distributed — Lumped — Discrete

where $r = \frac{D\Delta t}{h^2}$

$$U_i^{k+1} = U_i^k + r\delta_x^2 U_i^k$$

$$U_i^{k+1} = U_i^k + rU_{i-1}^k - 2rU_i^k + rU_{i+1}^k$$

$$U_i^{k+1} = rU_{i-1}^k (1-2r)U_i^k + rU_{i+1}^k$$

This is an explicit formulation for marching ahead in time.

How accurate is this approximation ?

- Use a Maximum principle analysis

Analytic soln. to $\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}$ is U

$$U_i^{k+1} = rU_{i-1}^k + \ 1 - 2r\ U_i^k + rU_{i+1}^k + 0(k+h^2)$$

↑ FD expression multi. by k

Subtract : $E_i^k = U_i^k - U_{Anal}$

$$E_i^{k+1} = rE_{i-1}^k + (1-2r)E_i^k + rE_{i+1}^k \quad \pm \quad \mathrm{O}(k+h^2)k$$

Parabolic IC's $\mathrm{E}_\mathrm{i}^0 = 0$

& for $\mathrm{r} < \frac{1}{2}$ $\quad 1 - 2\mathrm{r} > 0 \quad |1 - 2\mathrm{r}| = 1 - 2\mathrm{r}$

Then

$$\left|E_i^{k+1}\right| \le |r|\left|E_{i-1}^k\right| + |(1-2r)|\left|E_i^k\right| + |r|\left|E_{i+1}^k\right| \quad \pm \quad \left|\mathrm{O}(k+h^2)k\right|$$

for r < 1/2: $|\, r \,| + |\,(1-2r)\,| + |\, r \,| = 1$

$$\left\|E^{k+1}\right\| \le \left\|E^k\right\| \quad \pm \quad A(k+h^2)k$$

where "A" is an upper bound on the error.

but remember that $\|\mathrm{E}\|^0 = 0$

$$\left\|E^1\right\| < A(k+h^2)k \qquad \left\|E^2\right\| < A(k+h^2)2k$$

at a given point in time, $\left\|E^k\right\| < A(k+h^2)t$

Note that the value of A depends on upper bounds of

$$\frac{\partial^2 U}{\partial t^2} \quad and \quad \frac{\partial^4 U}{\partial x^4}$$

If as $t \to \infty$ U stabilizes the $\|E^k\| < A(k+h^2)t$

will vanish as A vanishes

What are the leading error terms? Recall Taylor's - Lecture 2

$$\frac{\partial U}{\partial t} = \frac{1}{\Delta t}\left[U^{k+1} - U^k\right] - \frac{\Delta t}{2!}\frac{\partial^2 U}{\partial t^2} - \frac{\Delta t^2}{3!}\frac{\partial^3 U}{\partial t^3} - \ldots h.o.t.$$

and

$$\frac{\partial^2 U}{\partial x^2} = \left[\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right] - \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4} \ldots h.o.t.$$

$$Akn(k + h^2)$$

$$kn\left[\frac{\Delta t}{2}\frac{\partial^2 U}{\partial t^2} = \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4}\right]$$

$$n\frac{k}{2}\left[k\frac{\partial^2 U}{\partial t^2} = \frac{h^2}{6}\frac{\partial^4 U}{\partial x^4}\right]$$

but $U_t = U_{xx}$ $\therefore$ $U_{tt} = U_{xxt} = U_{txx} = U_{xxxx}$

i.e U satisfies $U_{tt} = U_{xxxx}$

$$n\frac{k}{2}\left[k - \frac{h^2}{6}\right]\frac{\partial^4 U}{\partial x^4}$$ if k, h $\to 0$ s. t $\frac{k}{h^2} = \frac{1}{6}$

Then 1st leading errors in t, x cancel and letting $r = \frac{1}{6}$ accelerates convergence & obtains highly accurate solns

$$\left\| E^k \right\| < A(k^2 + h^4)t$$

If as $t \rightarrow \infty$ U stabilizes then

$\left\| E^k \right\| < A(k^2 + h^4)t$ will vanish rapidly as A vanishes

However, if the solution is periodic in time
(Ex: shallow water modeling of tidal cycles)

$\|E\|^k$ grows as time grows

Solution: transform from temporal to frequency domain or

let k and h spacing be adjusted in $\frac{1}{t}$ and $\frac{1}{\sqrt{x}}$ step sizes to maintain $(k + h^2)t$ constant. For long term simulations this is an excessive computational requirement

Recall also our wave Eqn.

$$\nabla^2 U - \frac{\partial^2 U}{\partial t^2} = 0$$ periodic soln $\Rightarrow$ $\nabla^2 U + \omega^2 U = 0$

Helmholtz Eqn which requires $\omega^2 h^2 - 4 > 4$ for d.d.

$\Rightarrow$ $h > .45L$

but 1% accuracy soln $\Rightarrow$ $h < .05L$

(Just reminding you that not all things can be solved transparently via numerical methods. Numerous fundamental obstacles exist in all fields of endeavor.)