Advective - Diffusive Equation (Steady - State)

SS : not accumulating or generating any more source effects

Conservation + Constitutive = PDE

$\nabla \cdot q = 0$ $\quad q = -D\nabla u + Vu$ $\quad D\nabla^2 u - V \cdot \nabla u = 0$

(D, V constant)

(V, D > 0)

$$1-D: \quad D\frac{d^2u}{dx^2} - V\frac{du}{dx} = 0$$

Dimensionless Form : $\chi = x/L$

$$\frac{\partial \chi}{\partial x} = \frac{1}{L} \Rightarrow \frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial \chi}\frac{\partial \chi}{\partial x} = \frac{\partial(\ )}{\partial \chi}\frac{1}{L}$$

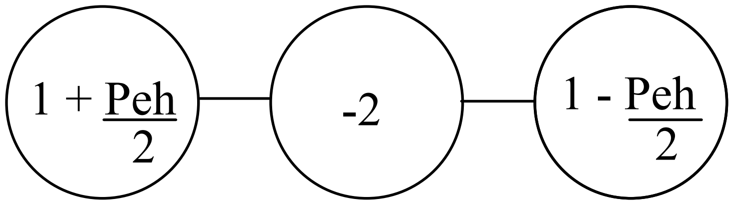
$$\frac{d^2u}{d\chi^2} - P_e\frac{du}{d\chi} = 0 \qquad P_e = \frac{VL}{D} \quad \text{"Peclet \#"}$$

ratio of advection to diffusion effects

FD form :

$$\frac{\delta_x^2 U_i}{h^2} - P_e(\ ?\ ) = 0 \qquad h = \Delta\chi = \Delta x/L$$

i) Centered : $P_e \dfrac{(U_{i+1} - U_{i-1})}{2h}$; multiplying the FD form by h^2:



= 0

Note: Diag. Dominance when $\dfrac{P_e h}{2} < 1$ i.e. $P_e h < 2$

(Weak Form)

A 2nd order linear constant coeff. soln => exp( )

Exact Solution of Difference Equations :

$$U_{i-1}\left[1 + \frac{P_e h}{2}\right] - 2U_i + U_{i+1}\left[1 - \frac{P_e h}{2}\right]$$

Try $U_i = \lambda^i$; solve for λ

$$U_i = \lambda U_{i-1}$$

$$U_{i+1} = \lambda^2 U_{i-1}$$

$$\left[1 + \frac{P_e h}{2}\right] - 2\lambda + \left[1 - \frac{P_e h}{2}\right]\lambda^2 = 0$$

$$\lambda = \frac{2 \pm \left[4 - 4\left[1 + \frac{P_e h}{2}\right]\left[1 - \frac{P_e h}{2}\right]\right]^{\frac{1}{2}}}{2\left[1 - \frac{P_e h}{2}\right]}$$

$$\lambda = \frac{1 \pm \left[1 - \left\{ 1 - \left(\frac{P_e h}{2} \right)^2 \right\} \right]^{\frac{1}{2}}}{1 - \frac{P_e h}{2}}$$

$$\lambda = \frac{1 \pm \frac{P_e h}{2}}{1 - \frac{P_e h}{2}}$$

$$\lambda_1 = 1 \quad \text{(Constant)}$$

$$\lambda_2 = \frac{\left[1 + \frac{P_e h}{2} \right]}{\left[1 - \frac{P_e h}{2} \right]}$$

$U_i = A + B\,\lambda_2^i$ Exact Soln to PDE :

$\lambda_2 \leftrightarrow \exp\,(P_e h)$ $U = C + D \exp\,(P_e \chi)$

The following soln satisfies the governing equation:

$$U = \frac{\exp\left(\frac{VL}{D} \right) - \exp\left(\frac{Vx}{D} \right)}{\exp\left(\frac{VL}{D} \right) - 1}$$

let $C = \dfrac{\exp\left(\frac{VL}{D}\right)}{\exp\left(\frac{VL}{D}\right) - 1}$ $\quad D = \dfrac{-1}{\exp\left(\frac{VL}{D}\right) - 1}$

$$U = C + D\exp\left(\frac{Vx}{D}\frac{L}{L}\right) = C + D\exp(P_e\chi)$$

$$U_i = A + B\lambda_2^i \quad \Leftrightarrow \quad C + D\exp(P_e\chi)$$

or

$$\lambda_2 \Leftrightarrow \exp(P_e h)$$
$$\lambda_2^i \leftrightarrow \exp(P_e i h) = \exp(P_e\chi)$$

$$\lambda_2 = \frac{1 + \frac{P_e h}{2}}{1 - \frac{P_e h}{2}} \qquad \text{let} \quad \xi = \frac{P_e h}{2}$$

$$\lambda_2 = (1+\xi)(1-\xi)^{-1}$$

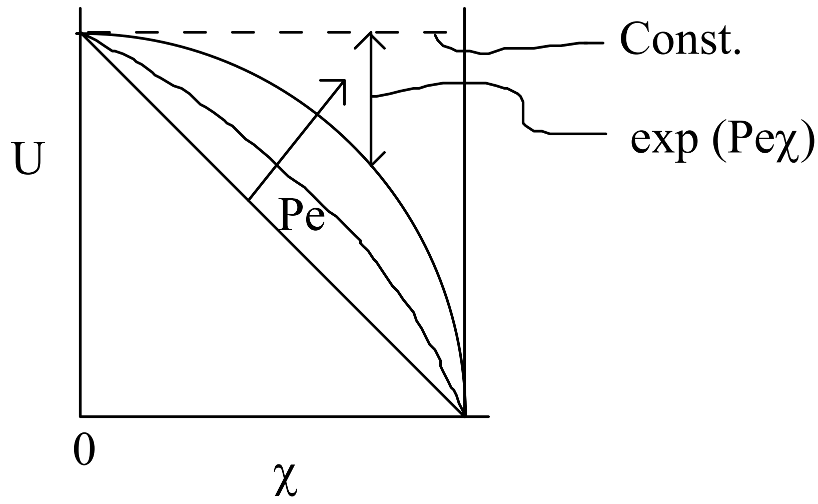
Binomial Expansion

Centered Formualtion gives:

$$\lambda_2 = (1+\xi)(1+\xi+\xi^2+\xi^3+\cdots) = 1 + 2\xi + 2\xi^2 + 2\xi^3 + \cdots$$

$$\exp(2\xi) = 1 + 2\xi + \frac{(2\xi)^2}{2!} + \frac{(2\xi)^3}{3!} + \cdots = 1 + 2\xi + 2\xi^2 + \frac{4}{3}\xi^3 + \cdots$$

Ex. BC's $\chi = 0$, $U = 1$
$\chi = 1$, $U = 0$

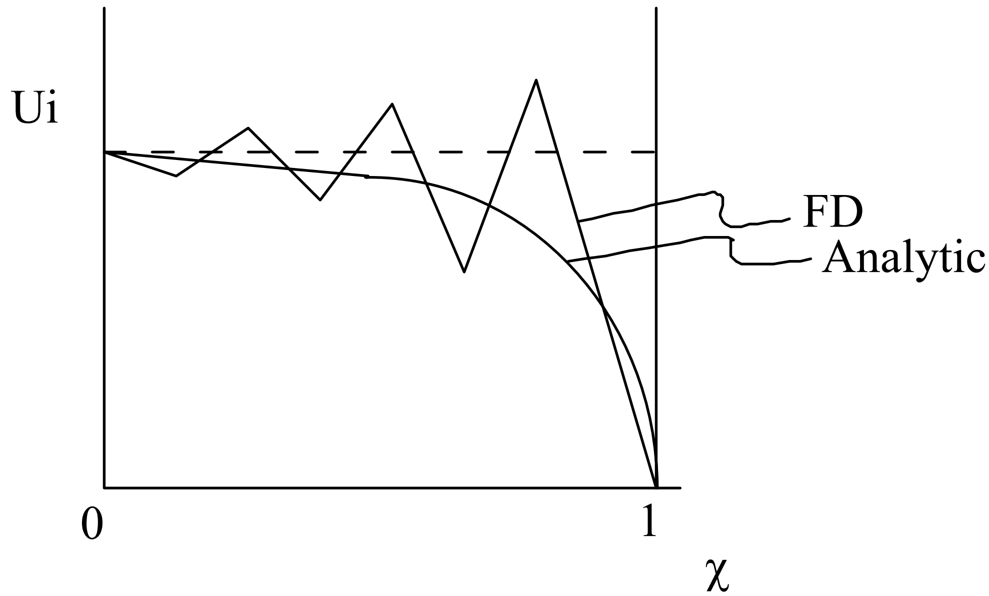


Numerical :

Accuracy depends on agreement between λ and exp $(P_e h)$

As $h \rightarrow 0$: convergence with 0(?) (student try it)

As $h \rightarrow$ large : λ becomes negative!



"Spurious Oscillation" when $\frac{Peh}{2} > 1$

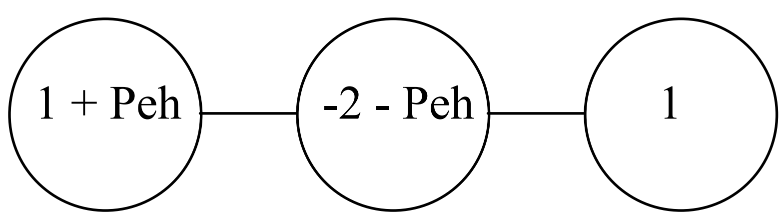
i.e. $Peh > 2$

"Cell Peclet #" $P_e h \equiv \frac{V \Delta x}{D}$

ii) Alternative : "Upstream Weighting" of $\frac{du}{d\chi}$

i.e. backward F.D.

$$\frac{\delta_x^2 U_i}{h^2} - P_e \left(\frac{U_i - U_{i-1}}{h} \right) = 0$$



= 0

Note: Always - (Weak) Diagonal Dominance

Difference Equations :

$$[1 + P_e h] - [2 + P_e h]\,\lambda + \lambda^2 = 0$$

$$\lambda = \frac{2 + P_e h \pm \sqrt{[2 + P_e h]^2 - 4[1 + P_e h]}}{2}$$

$$\lambda = \frac{2 + P_e h \pm P_e h}{2}$$

$\lambda_1 = 1$ (Constant) and $\lambda_2 = 1 + P_e h$ (Never Negative)

Or in terms of ξ: upstream $\lambda_2 = 1 + 2\xi$

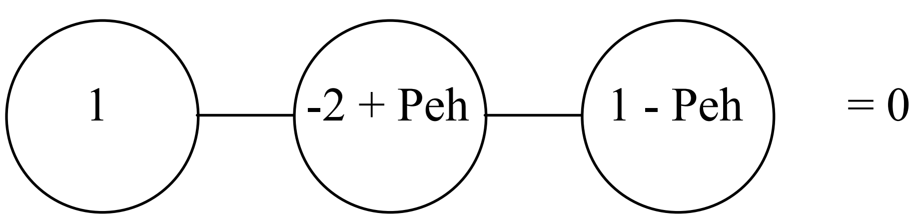
$\therefore$ Upstream Weighting $\rightarrow$ no spurious oscillation

Accuracy : λ_2 versus $\exp(P_e h)$

iii) Alternative : "Downstream Weighting" of $\frac{du}{d\chi}$

i.e forward Finite difference

$$\frac{\delta_x^2 U_i}{h^2} - P_e\left(\frac{U_{i+1} - U_i}{h}\right) = 0$$



$$1 + (-2 + P_e h)\,\lambda + (1 - P_e h)\,\lambda^2 = 0$$

$\lambda_1 = 1$ and $\lambda_2 = \dfrac{1}{1 - P_e h}$ --->

Same order approx. to $\exp(P_e h)$

as case ii (upstream weighting)

again in terms of ξ:

downstream $\lambda_2 = 1 + 2\xi + 4\xi^2 + 8\xi^3 + \cdots$

But : $P_e h > 1 \Rightarrow$ Oscillations

Summary:

Upstream	$P_e h < \infty$	; $0(h)$
Centered	$P_e h < 2$	; $0(h^2)$
Downstream	$P_e h < 1$	; $0(h)$

Wave Equation

$$\nabla^2 U - \frac{\partial^2 U}{\partial t^2} = 0$$

$\Uparrow$

Elliptic in Space

$\Uparrow$ $\Uparrow$

Hyperbolic Overall

$$U(x,t) = V(x) \cdot e^{j\omega t}$$

Periodic Solution: $\Uparrow$

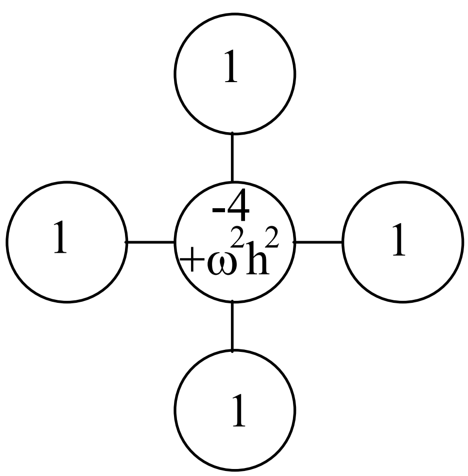
Amplitude

Helmholtz Equation : will always work for a linear hyperbolic eqn.

$$\nabla^2 V + \omega^2 V = 0 \quad \Leftrightarrow \quad \delta_x^2 V_{i,j} + \delta_y^2 V_{i,j} + \omega^2 h^2 V_{i,j} = 0$$

Elliptic

2 - D Molecule :



= 0

Diagonal Dominance:

- lost as $h \to 0$ $\therefore$ Cannot converge to PDE w/ D.D.
- available only when

$$\omega^2 h^2 - 4 > 4$$
$$\omega^2 h^2 > 8$$

Poor accuracy ; for ~ 1% , need $\omega h \le \frac{\pi}{10}$

$\left[\omega = \frac{2\pi}{L} \quad \text{so this is} \quad h \le \frac{L}{20} \right]$

Recall

$aU_{xx} + bU_{xy} + cU_{yy} + dU_x + eU_y + fU = g$

$aU_{xx} + cU_{yy} + fU = g$ and assumed $f \le 0$ & for $f < 0$

speed convergence

Telegraph Equation

$$\frac{\partial^2 U}{\partial t^2} + \tau \frac{\partial U}{\partial t} - c^2 \frac{\partial^2 U}{\partial x^2} = 0$$

τ is dampening or frictions factor which is necessary

It allows initial conditions to decay