Last Time

Point Iterative Methods

- At each step the approximate solution is modified at a single point of the domain.

- Each $U_{i,j}^{n+1}$ is determined Explicitly
  i.e. simultaneous solution of equations not required

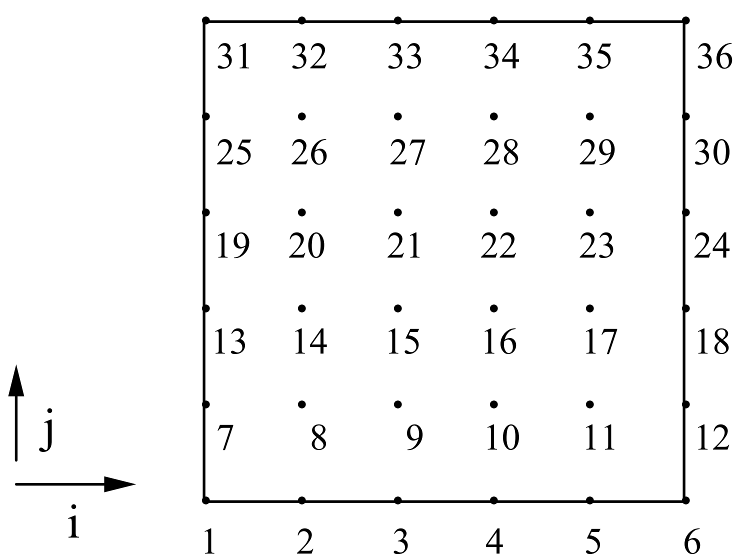
Today: Block Iterative Methods (/Implicit, /Group)

Generally, some level of implicitness leads to increased convergence rates.

All 3 point iterative methods can be converted to block iterative methods.

We will limit block groups to simple rows/columns

Consider the domain



with U specified on the boundary

The block (line) iterative method takes each row individually and writes an implicit 1-D formulation. For $0(h^2)$ accurate solutions of $\nabla^2 u = g$ a tridiagonal matrix developes exactly as in your first homework assignment.

Recall: $\nabla^2 u = g$

$$\begin{array}{ccc} & B_3 & \\ B_2 & B_0 & B_1 \\ & B_4 & \end{array} = h^2 g + B.C.'s$$

and for a point-wise Jacobi, Gauss-Seidel, and SOR iterations one has:

$$-\omega\Big[\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big] + (1-\omega)U^n_{i,j}$$

$$B_0U^{n+1}_{i,j} = -\left[\, B_1U^n_{i+1,j} + B_2\overset{n+1}{U^n_{i-1,j}} + B_3U^n_{i,j+1} + B_4\overset{n+1}{U^n_{i,j-1}} - Rhs \,\right]$$

For the Jacobi line-iterative system solve for all nodes along a row simultaneously using only U^n estimates on the RHS

$$\left[B_2U_{i-1,j} + B_0U_{i,j} + B_1U_{i+1,j}\right]^{n+1} = -\left[B_3U^n_{i,j+1} + B_4U^n_{i,j-1} - Rhs\right]$$

for all i = I1, I1 + 3
in our example

Specifically examine row 3 with nodes 13 -> 18

You write PDE on nodes 14 -> 17

G-S

*U_{13}

$$
\begin{array}{c}
B_2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} B_0 & B_1 & & \\ B_2 & B_0 & B_1 & \\ & B_2 & B_0 & B_1 \\ & & B_2 & B_0 \end{bmatrix}
\begin{array}{c}
\\ \\ \\ B_1
\end{array}
\begin{Bmatrix} U_{14} \\ U_{15} \\ U_{16} \\ U_{17} \end{Bmatrix}^{n+1}
=
\begin{Bmatrix}
-\left[B_3 U_{20}^{n} + B_4 \overset{\mathbf{n+1}}{U_{8}^{n}} - Rhs\right] \\
-\left[B_3 U_{21}^{n} + B_4 \overset{\mathbf{n+1}}{U_{9}^{n}} - Rhs\right] \\
-\left[B_3 U_{22}^{n} + B_4 \overset{\mathbf{n+1}}{U_{10}^{n}} - Rhs\right] \\
-\left[B_3 U_{23}^{n} + B_4 \overset{\mathbf{n+1}}{U_{11}^{n}} - Rhs\right]
\end{Bmatrix}
$$

*U_{18}

$U_{14} \rightarrow U_{17}$ are solved in Thomas routine
Then move to row 4 etc.

The Gauss-Seidel only modifies the $U_{i,j-1}$ values since these n+1 estimates were just obtained. Note that the use of $U_{i,j-1}^{n+1}$ is still an explicit calculation.

The S.O.R formulation of row 3 is

$$\begin{bmatrix} B_0 & \omega B_1 & & \\ \omega B_2 & B_0 & \omega B_1 & \\ & \omega B_2 & B_0 & \omega B_1 \\ & & \omega B_2 & B_0 \end{bmatrix} \begin{Bmatrix} U_{14} \\ U_{15} \\ U_{16} \\ U_{17} \end{Bmatrix}^{n+1} =$$

$$- \omega B_3 \begin{Bmatrix} U_{20} \\ U_{21} \\ U_{22} \\ U_{23} \end{Bmatrix}^{n} - \omega B_4 \begin{Bmatrix} U_{8} \\ U_{9} \\ U_{10} \\ U_{11} \end{Bmatrix}^{n+1} - \omega \begin{Bmatrix} RHS \end{Bmatrix} +$$

$$(1 - \omega) B_0 \begin{Bmatrix} U_{14} \\ U_{15} \\ U_{16} \\ U_{17} \end{Bmatrix}^{n} - \omega B_2 U_{13} - \omega B_1 U_{18}$$

As shown last time

# of iterations $\quad M = \dfrac{-\ln(\kappa)}{-\ln(\rho)}$ error reduction factor

Convergence Rate $\quad R(G) = -\ln(\rho)$

	Point	Line	Improvement
Jacobi	$\frac{1}{2}h^2$	h^2	2* or $\equiv$ GS Point
G.-S.	h^2	$2h^2$	2*
S.O.R.	$2h$	$\sqrt{2}\ 2h$	$\sqrt{2}$

Therefore, a small increase in coding (i.e. use Thomas algorithm) doubles output rate.

The strategy of solving line by line could have been column by column with comparable results. It seems intuitive then that solving the system implicitly first along the rows then implicitly along the columns might improve the convergence.

This is the basis of the Alternating Direction Implicit iteration method, ADI.

ref.

Birkholl, G., Varga, R.S. and Young, D., "Alternating Direction Implicit Methods", Advances in Computers, F.L. Alt and M. Rubinoff, eds, pg. 189-273, Academic Press, N.Y. 1962.

Although this reference is an extensive survey of ADI, the theory behind the convergence of ADI routines is still lacking.

However, from an engineering standpoint ADI works.

Consider the following Self-Adjoint Elliptic Equation

$$\nabla^2 u + fu = g \qquad \text{with } f \le 0$$

The difference formulation is

$$\boxed{\mathbf{-\ \omega U_{ij}^{n+1}}}\ \delta_x^2 U_{ij} + \delta_y^2 U_{ij} + fh^2 U_{ij} = gh^2 \boxed{\mathbf{-\ \omega U_{ij}^{n}}}$$

(assume $\Delta x = \Delta y \Rightarrow \beta = 1$)

The ADI method introduces an iteration parameter ω to the PDE, (some text use ρ symbol - not spectral radius)

This procedure is a two step process - Implicit in X followed by Implicit in Y

Step 1: Implicit in x.

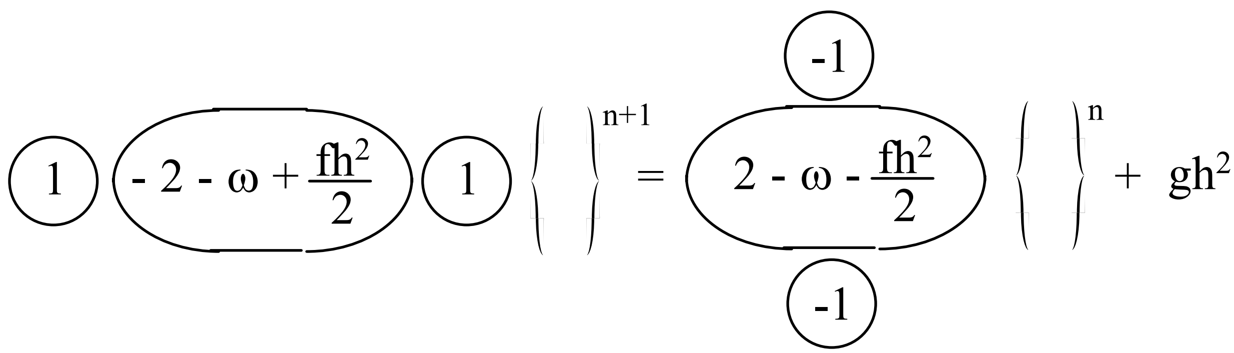
$$- \omega U_{ij}^{n+1} + \left(\delta_x^2 + \frac{fh^2}{2}\right) U_{ij}^{n+1} = - \left(\delta_y^2 + \frac{fh^2}{2}\right) U_{ij}^{n} - \omega U_{ij}^{n} + gh^2$$

Step 2: Implicit in y.

$$- \omega U_{ij}^{n+2} + \left(\delta_y^2 + \frac{fh^2}{2}\right) U_{ij}^{n+2} = - \left(\delta_x^2 + \frac{fh^2}{2}\right) U_{ij}^{n+1} - \omega U_{ij}^{n+1} + gh^2$$

This 2 step procedure has a tridiagonal matrix on LHS and one checks for convergence on even iteration counts.

The Tridiagonal computational molecule is



Using Iteration matrix notation

$[A]\{U\} = \{V\}$

Let $\underset{}{[A]} = \underset{\delta_x^2}{[X]} + \underset{\delta_y^2}{[Y]} + \underset{f}{[F]}$

Then

Step 1:

$$\left[X + \tfrac{1}{2}F - \omega I\right]\{U\}^{n+1} = \left[- Y - \tfrac{1}{2}F - \omega I\right]\{U\}^{n} + \{V\}$$

Step 2:

$$\left[Y + \tfrac{1}{2}F - \omega I\right]\{U\}^{n+2} = \left[- X - \tfrac{1}{2}F - \omega I\right]\{U\}^{n+1} + \{V\}$$

$\Rightarrow$

$$\{U\}^{n+1} = \left[X + \tfrac{1}{2}F - \omega I\right]^{-1}\left(\left[- Y - \tfrac{1}{2}F - \omega I\right]\{U\}^{n} + \{V\}\right)$$

$$\{U\}^{n+2} = \left[Y + \tfrac{1}{2}F - \omega I\right]^{-1}\left(\left[- X - \tfrac{1}{2}F - \omega I\right]\{U\}^{n+1} + \{V\}\right)$$

$$\{U\}^{n+2} = \left[Y + \tfrac{1}{2}F - \omega I\right]^{-1}\left[- X - \tfrac{1}{2}F - \omega I\right]\left[X + \tfrac{1}{2}F - \omega I\right]^{-1}\left[- Y - \tfrac{1}{2}F - \omega I\right]\{U\}^{n}$$

$$+ \left[Y + \tfrac{1}{2}F - \omega I\right]^{-1}\Big[\, I + \left[- X - \tfrac{1}{2}F - \omega I\right]\left[X + \tfrac{1}{2}F - \omega I\right]^{-1}\Big]\{V\}$$

For the problem $\nabla^2 u - fu = g$ on a perfect rectangle one has an optimum ω as:

$$\omega_{/opt} = \left\{\left[- \frac{1}{2}h^2 f + 4\sin^2\left(\frac{\pi}{2R}\right)\right]\left[- \frac{1}{2}h^2 f + 4\cos^2\left(\frac{\pi}{2R}\right)\right]\right\}^{1/2}$$

R = max (#Row, #Columns) WITH Type I BCs all around