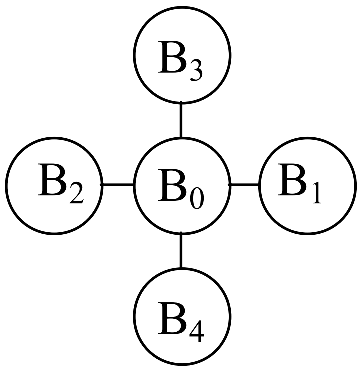
Last Time - Point Iterative Methods

$$[A]\{U\} = \{v\} \quad \text{for the system} \quad \nabla^2 U = g$$

with a computational molecule



$= h^2 g + \text{B.C.s}$

[A] partitioned into [R] + [D] + [S]
Below Diagonal Above

ω **n** **n+1** **n** **n+1** $\mathbf{+(1-\omega)U^{n}_{i,j}}$

n *n*+1 *n* *n*+1

$$U_{i,j}^{n+1} = \frac{-1}{B_0}\left[B_1 U^{n}_{i+1,j} + B_2 U^{n}_{i-1,j} + B_3 U^{n}_{i,j+1} + B_4 U^{n}_{i,j-1} - \text{Rhs}\right]$$

Jacobi:

$$\{U\}^{n+1} = \underbrace{-[D]^{-1}[R+S]}_{[G_J]}\{U\}^{n} + [D]^{-1}\{V\}$$

Gauss - Seidel:

$$\{U\}^{n+1} = \underbrace{-[R+D]^{-1}[S]}_{[G_{GS}]}\{U\}^{n} + [R+D]^{-1}\{V\}$$

S.O.R.:

$$\{\mathbf{U}\}^{\mathbf{n+1}} = \underbrace{[\mathbf{D} + \omega\mathbf{R}]^{-1}[(1 - \omega)\mathbf{D} - \omega\mathbf{S}]}_{[\mathbf{G}_\omega]}\{\mathbf{U}\}^{\mathbf{n}} + [\mathbf{D} + \omega\mathbf{R}]^{-1}\omega\{\mathbf{V}\}$$

Basic Rule:

Type I Boundary: Do not use the PDE on the Bdy.

Type II or III Bdys: Use PDE <u>Plus</u> B.C. together

Spectral Radius, ρ, of iteration matrix [G] is the largest magnitude eigenvalue of [G]

$\rho < 1$ for convergence

Bare Essentials of Iterative Methods

Computational Estimate for ρ

$$\{\delta\}^n = \{U\}^n - \{U\}^{n-1}$$

$$\rho \cong \frac{\|\delta^n\|}{\|\delta^{n-1}\|} = \frac{\left[\sum_{i=1}^{M}\left(U_i^n - U_i^{n-1}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{M}\left(U_i^{n-1} - U_i^{n-2}\right)^2\right]^{1/2}}$$

Now to prove that an iteration scheme can converge

Consider the following worst case situations.
Recall def. Strict Diagonal Dominance

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Expanding on the iteration handout

Define $\{\varepsilon\}^n = \{U\}^n - \{U\} = \{U\}^n - [A]^{-1}\{v\}$

↑ exact algebraic solutions (unknown)

Since

$$\{U\}^n = [G]\{U\}^{n-1} + \{r\} \qquad \text{and} \qquad \{U\} = [G]\{U\} + \{r\}$$

$$\{\varepsilon\}^n = [G]\{U\}^{n-1} + \{r\} - G\{U\} - \{r\} = [G](\{U\}^{n-1} - \{U\})$$

or

$$\{\varepsilon\}^n = [G]\{\varepsilon\}^{n-1} = [G]([G]\{\varepsilon\}^{n-2}) \Rightarrow [G]^n\{\varepsilon\}^0$$

However we still don't know $\{\varepsilon\}^0$
But

Define $\{\delta\}^n = \{U\}^n - \{U\}^{n-1}$ This incremental error can be determined for all n

$$\{\delta\}^n = [G]\{U\}^{n-1} + \{r\} - [G]\{U\}^{n-2} - \{r\} = [G](\{U\}^{n-1} - \{U\}^{n-2})$$

or

$$\{\delta\}^n = [G]\{\delta\}^{n-1} \Rightarrow [G]^n\{\delta\}^0$$

Finally we can examine Residuals:

Normally $[A]\{U\} = \{v\}$

$$0 = [A]\{U\} - \{v\} \neq [A]\{U\}^n - \{v\}$$

Define

$$\begin{aligned}\{R\}^n &= [A]\{U\}^n - \{v\} \\ &= [A]\{U\}^n - [A][A]^{-1}\{v\} \\ &= [A](\{U\}^n - [A]^{-1}\{v\})\end{aligned}$$

remember $\{\varepsilon\}^n = \{U\}^n - \{U\}$

$$\begin{aligned}\{R\}^n &= [A]\{\varepsilon\}^n \\ &= [A][G]\{\varepsilon\}^{n-1} = [A][G]\,[A]^{-1}\underbrace{[A]\{\varepsilon\}^{n-1}}_{\{R\}^{n-1}}\end{aligned}$$

$$\{R\}^n = [A][G][A]^{-1}\{R\}^{n-1} \Rightarrow [A][G]^n[A]^{-1}\{R\}^0$$

Therefore, we have the following error measures

$\{\varepsilon\}^n = [G]\{\varepsilon\}^{n-1} \Rightarrow [G]^n\{\varepsilon\}^0$ numerical vs algebraic

$\{\delta\}^n = [G]\{\delta\}^{n-1} \Rightarrow [G]^n\{\delta\}^0$ incremental errors

$\{R\}^n = [G]\{R\}^{n-1} \Rightarrow [G]^n\{R\}^0$ residual error

Each of these error indicators converges to zero if and only if the spectral radius, ρ, (or largest absolute value eigenvalue) of the iteration matrix is less than 1.

Therefore,

$$\{\varepsilon\}^n \cong \rho\{\varepsilon\}^{n-1}$$

$$\{\delta\}^n \cong \rho\{\delta\}^{n-1}$$

$$\{R\}^n \cong \rho\{R\}^{n-1}$$

and one can estimate the spectral radius of the system via

$$\rho \cong \frac{\left\|\delta^n\right\|}{\left\|\delta^{n-1}\right\|} = \frac{\left[\sum_{i=1}^{M}\left(U_i^n - U_i^{n-1}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{M}\left(U_i^{n-1} - U_i^{n-2}\right)^2\right]^{1/2}}$$

If one measures ρ expect the following

Given

$$[A]\{U\} = \{v\}$$ where [A] strict diagonal dominance

Prove Convergence

Define $$\theta_i = \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|}$$

Recall $$\{\varepsilon\}^{n+1} = [G]\{\varepsilon\}^n$$

Jacobi $$[G_J] = -[D]^{-1}[R+S] = -\frac{\sum_{j \neq 1} a_{ij}}{a_{ii}}$$

$$\varepsilon \ : \ \varepsilon_i^{n+1} = \frac{1}{a_{ii}}\left[-\sum_{j \neq 1} a_{ij}\varepsilon_j^n\right]$$

$$\left|\varepsilon_i^{n+1}\right| \le \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|}\left|\varepsilon_j^n\right| \le \theta_i \left\|\varepsilon\right\|^n$$

where $\|\varepsilon\|^n \quad \max_j |\varepsilon_j|^n$

Worst case

$$\|\varepsilon\|^{n+1} \le \theta_{max} \|\varepsilon\|^n$$

$\Rightarrow \ \theta_{max} < 1$ sufficient for convergence

Note:

Elliptic equation $\theta_{max} = 1$ $\therefore$ Jacobi will not diverge.

Examine Gauss - Seidel :

$$\varepsilon_1^{n+1} = \frac{-1}{a_{11}} \sum_{j=2}^{N} a_{1j}\varepsilon_j^n \Rightarrow \left|\varepsilon_1\right|^{n+1} \leq \theta_1 \left\|\varepsilon\right\|^n \leq \left\|\varepsilon\right\|^n$$

since $\theta_1 < 1$

$$\varepsilon_2^{n+1} = \frac{-1}{a_{22}} \left[a_{21}\varepsilon_1^{n+1} + \sum_{j=3}^{N} a_{2j}\varepsilon_j^n \right]$$

$$\left|\varepsilon_2\right|^{n+1} \leq \frac{|a_{21}|}{|a_{22}|} \|\varepsilon\|^n + \sum_{j=3}^{N} \frac{|a_{2j}|}{|a_{22}|} |\varepsilon_j|^n \leq \theta_2 \|\varepsilon\|^n$$

etc. . . .

In general since $\frac{|a_{21}|}{|a_{22}|} \|\varepsilon_1\|^{n+1} < \frac{|a_{21}|}{|a_{22}|} \|\varepsilon\|^n$

the Gauss - Seidel will converge faster (or diverge faster) than Jacobi.

and

$$\rho_{GS} = \rho_J^2$$

From S.O.R. theory for [A] Symmetric, Consistently ordered, "Property A"

$$\rho_{GS} = \rho_J^2$$

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho_J^2}} = \frac{2}{1 + \sqrt{1 - \rho_{GS}}}$$

Recall: Self Adjoint implies symmetry

Rates of Convergence

In the limit of large n, recall that

$\{\varepsilon\}^{n+M} = \rho^M \{\varepsilon\}^n$ and therefore

$$\|\varepsilon\|^{n+M} = \rho^M \|\varepsilon\|^n$$

or

$$\rho^M = \frac{\|\varepsilon\|^{n+M}}{\|\varepsilon\|^n}$$

and if you wish to reduce the existing error by a factor of K

$$K = \frac{\|\varepsilon\|^{n+M}}{\|\varepsilon\|^n}$$

one can write

$\rho^M = K$ and solve for M, the number of iterations required to get to desired accuracy

$$M = \ln(K) / \ln(\rho)$$

When solving $\nabla^2 U = 0$ on a square

Jacobi

$$\rho = \max|\lambda| = \cos h \sim 1 - \frac{h^2}{2} \text{ as } h \rightarrow 0$$

The rate of convergence of a linear iteration

$\{U\}^{n+1} = [G]\{U\}^n + \{r\}$ characterized by the matrix [G] is

$$R(G_J) = - \log \lambda(G_J) = - \log \rho$$

(-) since $\rho < 1$ for (+) convergence rate

Ref:

Young, D.M. Trans. AM. Math. Soc., 76, #92, 1954
Ames - on reserve
Westlake - listed in handout class 1 - Appendix B
Eigenvalue Bounds

and $- \log \rho \sim - \log\left(1 - \frac{h^2}{2}\right) = \frac{h^2}{2} + O(h^4)$

Thus, the convergence rate for Jacobi iterations is approximately $h^2/2$ which is slow for small values of h

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \qquad -1 < x \leq 1$$

The max $|\lambda|$ in $[G_{GS}]$ is $\cos^2 h \sim 1 - h^2$ as $h \to 0$

$$\therefore \quad R(G_{GS}) = -\log(\cos^2 h) \sim h^2 + O(h^4)$$

or the Gauss-Seidel iterations will converge twice as fast as Jacobi.

Finally

$$\max|\lambda| \text{ in } [G_\omega] \sim 1 - 2h \quad \text{as } h \to 0$$

$$R(G) \sim 2h + O(h^2) \quad \text{for optimal S.O.R.}$$

$\dfrac{2h}{h^2} \sim \dfrac{2}{h}$ times faster than Gauss-Seidel which for small h is significant