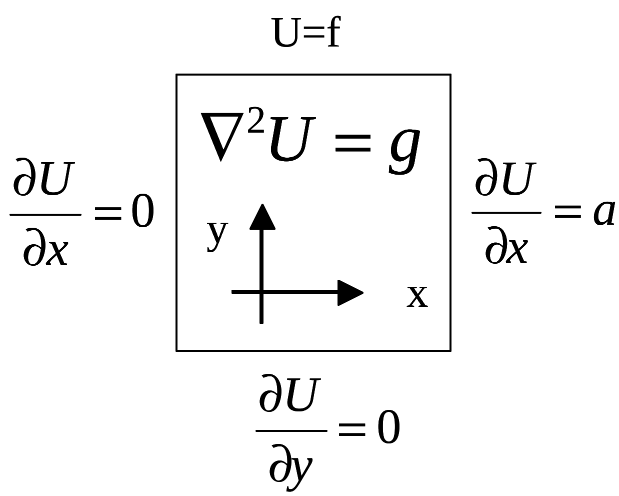
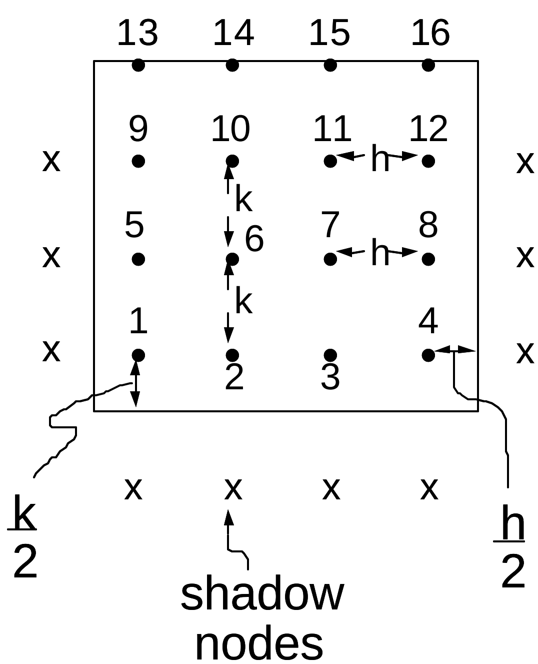
Last time we examined Poisson's Equation on a rectangle

PDE: $$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = g$$

with the following boundary conditions:



We discretized the domain as follows:



The computational molecule that we apply to the discretized region is based on the finite difference expression of the PDE.

$$
\begin{array}{ccccc}
 & & & \frac{1}{k^2} & \\
\frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & + \; \frac{-2}{k^2} & = g \\
 & & & \frac{1}{k^2} &
\end{array}
$$

The interior computational molecule - valid at all interior nodes

$$
\begin{array}{ccccc}
 & \beta & & & \\
1 & -2(1+\beta) & 1 & = & gh^2 \\
 & \beta & & &
\end{array}
$$

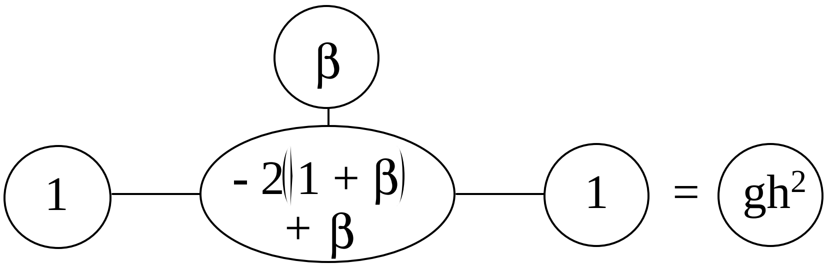
Where $\beta = \frac{h^2}{k^2}$

Left boundary

$\frac{\partial U}{\partial x} = 0$ x  $\bullet$ $\frac{u_{\bullet} - u_x}{h} = 0 \Rightarrow$

$$
\begin{array}{cccc}
\beta & & & \\
\begin{array}{c} -2(1+\beta) \\ +1 \end{array} & 1 & = & gh^2 \\
\beta & & &
\end{array}
$$

Bottom boundary

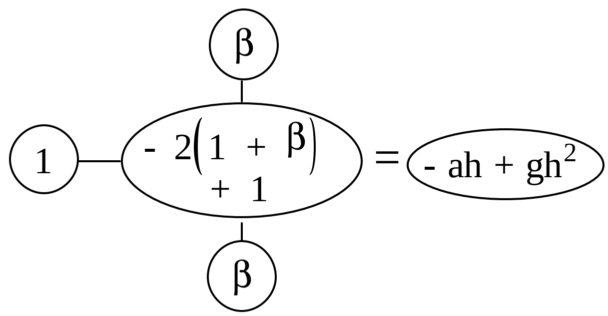


Right boundary

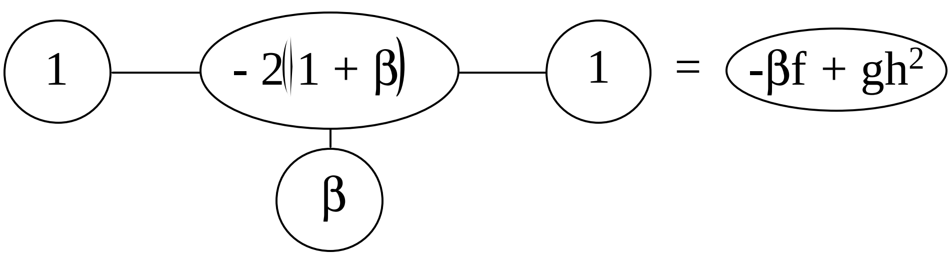
• x

$$\frac{u_x - u_\bullet}{h} = a$$

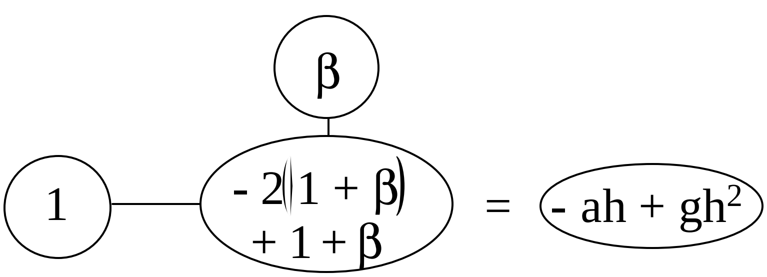
$$u_x = u_\bullet + ah$$



Top boundary (nodes 9-12)



Corner node 4



Basic Rule:

Type I Boundary : Do not use the PDE

Type II or III Boundary : Use PDE plus B.C. together

Solution Strategies

A.) Direct :

Exact algebraic solution in finite # of steps

non-repetitive

Complicated coding

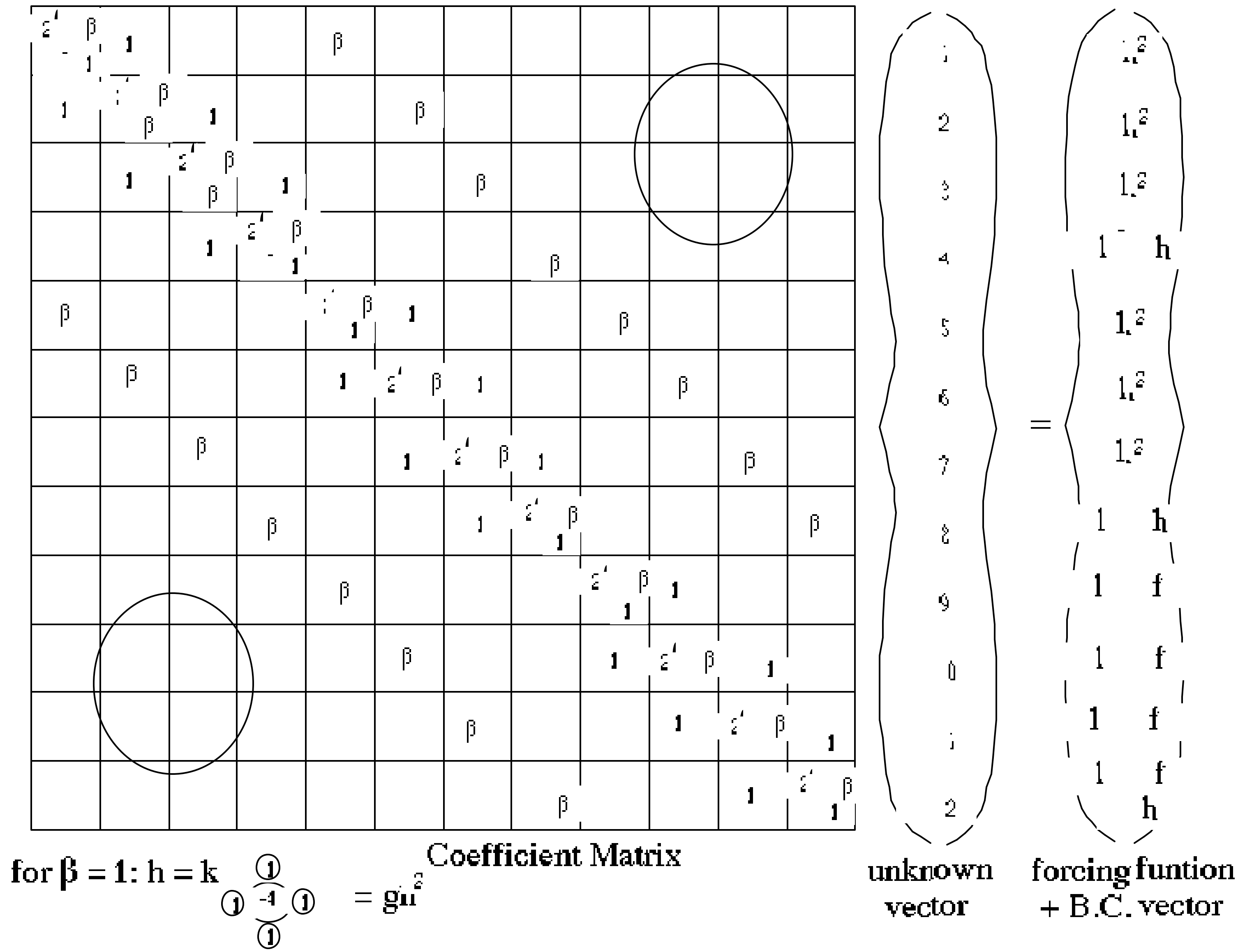
Exploit sparse / banded structure

Node numbering dictates the band structure

LU Decomposition is popular

- preserves the bandwidth
- The back substitution step is fast relative to the decomposition step. Therefore numerous solutions to the same coefficient matrix can be gotten cheaply.

For this example, the following matrix would be constructed:



Coefficient Matrix
unknown vector
forcing funtion + B.C. vector
for β = 1: h = k
-4
= gh²

B.) Iterative Solution Strategies

- Exact algebraic solutions only after ∞ # of solutions
- Monotonously repetitive
- Coding simple; proceeds directly from <u>Molecule</u>
- Banded structure is irrelevant
- Retain double subscripts $U_{i,j}$
- Node numbering is important
    - Dictates the order in which iteration proceeds
    - $\therefore$ Can determine convergence properties
- "Point" versus "Block or Line" methods

Point Iterative Methods

Elliptic Equation

$$\frac{\partial}{\partial x}\left(a\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial U}{\partial y}\right)+d\frac{\partial U}{\partial x}+e\frac{\partial U}{\partial y}+fU=g$$

b = 0 the coefficient of $\frac{\partial^2 U}{\partial x \partial y}$; a > 0, c > 0; f $\le$ 0 the f statement adds stability (more later on that topic)

Consider the previous Poisson's Equation

$$[A]\{u\} = \{v\}$$

$h^2 g$ + B.C.'s

let the coefficients in the computational molecule be

$$\begin{array}{ccc} & B_3 & \\ B_2 & B_0 & B_1 \\ & B_4 & \end{array}$$

A has diagonal dominance

$$\begin{cases} \alpha_{ii} > 0 \qquad \alpha_{ij} \le 0 \quad i \ne j \\ \alpha_{ii} \ge \sum_{j=i} |\alpha_{ij}| \ \text{; strict inequality for some i} \end{cases}$$

A is symmetric $\alpha_{ij} = \alpha_{ji}$ and self-adjoint d = e = 0

irreducible when all V_i impact all U_i

The Jacobi point iterative method is :

$$U_{i,j}^{n+1} = -\frac{1}{B_0}\left[B_1 U_{i+1,j} + B_2 U_{i-1,j} + B_3 U_{i,j+1} + B_4 U_{i,j-1} - \left(gh^2 + B.C.\right)\right]^n$$

where n is the iteration index

Partitions $[A] \Rightarrow [R] + [D] + [S]$

below diagonal above

Then $[A]\{u\} = \{v\}$

$$[R]\{u\}^n + [D]\{u\}^{n+1} + [S]\{u\}^n = \{v\}$$

$$\{u\}^{n+1} = -\underbrace{\left[D^{-1}\right]\left[R + S\right]}_{\text{Jacobi Iteration Matrix } [G_J]}\{u\}^n + \left[D^{-1}\right]\{v\}$$

The Jacobi iterative method expresses the n+1 values exclusively in terms of the iteration values.

The Gauss - Seidel Iterative method uses the n+1 iteration values as soon as possible, i.e.

$$U_{i,j}^{n+1} = -\frac{1}{B_0}\left[B_1 U_{i+1,j}^{n} + B_2 U_{i-1,j}^{n+1} + B_3 U_{i,j+1}^{n} + B_4 U_{i,j-1}^{n+1} - \left(h^2 g + B.C.\right)\right]$$

or

$$[R + D]\{u\}^{n+1} = -[S]\{u\}^{n} + \{v\}$$

with Gauss Iteration Matrix

$$[G_G] = -[R + D]^{-1}[S]$$

The SOR iteration, successive over-relaxation method introduces an acceleration (or relaxation) factor ω where

$0 < \omega < 2$. For $\omega = 1$ the SOR method $\equiv$ Gauss-Seidel method.

$$U_{ij}^{n+1} = -\frac{\omega}{B_0}\underbrace{\left[B_1 U_{i+1,j}^{n} + B_2 U_{i-1,j}^{n+1} + B_3 U_{i,j+1}^{n} + B_4 U_{i,j-1}^{n+1} - \left(h^2 g + B.C.\right)\right]}_{\text{Gauss Seidel Estimate}} + (1 - \omega)U_{i,j}^{n}$$

$$[D + \omega R]\{U\}^{n+1} = [(1 - \omega)D - \omega S]\{U\}^{n} + \omega\{V\}$$

With the SOR iterations matrix defined as:

$$[G_\omega] = [D + \omega R]^{-1}[(1 - \omega)D - \omega S]$$

These iteration methods will converge if and only if the spectral radius of G < 1

$$\rho(G) < 1 \quad \text{for convergence}$$

where ρ is the largest (abs. value) eigenvalue of [G]

If [A] is diagonally Dominant the J and G-S Converge.

is Symmetric and Positive Definite $\Rightarrow \{x\}^T[A]\{x\} > 0$

then J,G-S, and SOR converge for $0 < \omega < 2$

Generally: If J converges then G-S converges faster and SOR will be optional for some ω.

Each of the iterative methods can be written as

$$\{U^{n+1}\} = [G]\{U^n\} + \{v\}$$

The exact analytic solution to the <u>algebraic</u> Eqn

$$\{U_A\} = [G]\{U_A\} + \{v\}$$

The error e^k in the k^{th} iterations is

$$e^k = U_A - U^n$$

$$\{U_A\} - \{U^{n+1}\} = [G]\{U_A - U^n\} + 0$$

$$e^{k+1} = [G]\, e^k$$

In general

$$e^k = \left.\begin{matrix}[G]\, e^{k-1} \\ [G]\, e^{k-2}\end{matrix}\right\} [G]^2\, e^{k-2}$$

$$e^k = [G]^m\, e^{k-m}$$

$$e^k = [G]^k\, e^0$$

The system will converge iff

$$\lim_{k\to\infty} e^k = 0$$

since U initial, hence e^0 is arbitrary

$$\lim_{k\to\infty} [G]^k = \underline{0}$$

BARE ESSENTIALS OF ITERATIVE METHODS

Lynch, D.R., '83

Let us express the collection of FD equations as

$$[A]\{u\} = \{v\} \tag{1}$$

We wish to solve this by an iterative method

$$\{u^n\} = [G]\{u\}^{n-1} + \{r\} \tag{2}$$

Where [G] is the *iteration matrix*. Since we require that $\{u\} = [G]\{u\} + \{r\}$, and substituting $\{u\} = [A^{-1}]\{v\}$, we obtain the requirement that

$$\{r\} = [I - G][A^{-1}]\{v\} \tag{3}$$

a) Errors. We define the error vector at the end of iteration n to be

$$\{\varepsilon\}^n = \{u\}^n - \{u\} = \{u\}^n - [A^{-1}]\{v\} \tag{4}$$

Since

$$\{u\}^n = [G]\{u\}^{n-1} + \{r\} \tag{5}$$

and

$$\{u\} = [G]\{u\} + \{r\} \tag{6}$$

we obtain by subtraction the recursion relation for the errors:

$$\{\varepsilon\}^{n} = [G]\{\varepsilon\}^{n-1} \qquad (7)$$

or equivalently

$$\{\varepsilon\}^{n} = [G]^{n}\{\varepsilon\}^{0} \qquad (8)$$

This relationship is utilized in proofs of convergence, etc.; in computational practice it cannot be used since by hypothesis the array {u} (the exact algebraic solution to the FD equations) is unknown. Note that $\{\varepsilon\}^{n}$ as defined here is *unrelated* to the differences between the exact analytic solution and the exact algebraic FD solution.

b) Increments. The increment to the solution vector which occurs during the n^th iteration is

$$\{\delta\}^{n} = \{u\}^{n} - \{u\}^{n-1} \qquad (9)$$

Writing equation (2) for $\{u\}^{n}$ and also for $\{u\}^{n-1}$

$$\{u\}^{n} = [G]\{u\}^{n-1} + \{r\}$$
$$\{u\}^{n-1} = [G]\{u\}^{n-2} + \{r\}$$

and subtracting yields the recursion relation for the increments:

$$\{u\}^{n} - \{u\}^{n-1} = [G]\left(\{u\}^{n-1} - \{u\}^{n-2}\right) + \left(\{r\} - \{r\}\right)$$

or

$$\{\delta\}^n = [G]\{\delta\}^{n-1} \tag{10}$$

While the errors $\{\varepsilon\}$ cannot be measured, the $\{\delta\}$ can and thus provide useful computational information on the progress of an iterative method.

c) Residuals. Another measurable factor is the residual of the algebraic system after the n^{th} iteration; which must vanish as $n \rightarrow \infty$:

$$\{R\}^n = [A]\{u\}^n - \{v\} \tag{11}$$

or, taking advantage of equation (4),

$$\{R\}^n = [A]\left(\{u\}^n - [A^{-1}]\{v\}\right) = [A]\{\varepsilon\}^n \tag{12}$$

Use of the recursions for $\{\varepsilon\}^n$, eq. (7,8), yields

$$\{R\}^n = [A][G]\{\varepsilon\}^{n-1} = [A][G][A^{-1}]\{R\}^{n-1} \tag{13}$$

and

$$\{R\}^n = [A][G]^n[A^{-1}]\{R\}^0 \tag{14}$$

d) Convergence. The iteration (2) converges if and only if the spectral radius ρ of the iteration matrix [G] is less than unity. By definition, ρ is the largest (in absolute value) eigenvalue of

[G]. Noting that the eigenvalues of $[A][G][A^{-1}]$ are the same as those of [G], we have in the limit of the large n:

$$\{\varepsilon\}^n \cong \rho\{\varepsilon\}^{n-1}$$

$$\{\delta\}^n \cong \rho\{\delta\}^{n-1}$$

$$\{R\}^n \cong \rho\{R\}^{n-1}$$

The second and third of these are useful in estimating ρ. For example,

$$\rho \cong \frac{\|\delta^n\|}{\|\delta^{n-1}\|}$$

where the $\|\delta\|$ notation indicates the Euclidean norm (length) of a vector of length M:

$$\|d\| = \left(\sum_{i=1}^{M} d^2\right)^{1/2}$$

Other norms of the form

$$\|d\| = \left(\sum_{i=1}^{M} |d|^{N} \right)^{1/N}$$

are also useful, the most common being N = 1 and the limiting case N = ∞;

$$\|\delta\| = \max_{i} |\delta_i|$$