Robust XVA

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Abstract

We introduce an arbitrage-free framework for robust valuation adjustments. An investor trades a credit default swap portfolio with a risky counterparty, and hedges credit risk by taking a position in the counterparty bond. The investor does not know the expected rate of return of the counterparty bond, but he is confident that it lies within an uncertainty interval. We derive both upper and lower bounds for the XVA process of the portfolio, and show that these bounds may be recovered as solutions of nonlinear ordinary differential equations. The presence of collateralization and closeout payoffs leads to fundamental differences with respect to classical credit risk valuation. The value of the super-replicating portfolio cannot be directly obtained by plugging one of the extremes of the uncertainty interval in the valuation equation, but rather depends on the relation between the XVA replicating portfolio and the close-out value throughout the life of the transaction.

Keywords: robust XVA, counterparty credit risk, backward stochastic differential equation, arbitrage-free valuation.

Mathematics Subject Classification (2010): 91G40, 91G20, 60H10

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1 Introduction

Dealers need to account for market inefficiencies related to funding and credit valuation adjustments when marking their swap books. Those include the capital needed to support the trading position, the losses originating in case of a premature default by either of the trading parties, and the remuneration of funding and collateral accounts. It is common market practice to refer to these costs as the XVA of the trade. Starting from 2011, major dealer banks have started to mark these valuation adjustments on their balance sheets; see, for instance, Cameron (2014) and Beker (2015).

A large body of literature has studied the implication of these costs on the valuation and hedging of derivatives positions. Crépey (2015a) and Crépey (2015b) use backward stochastic differential equations to value the transaction, accounting for funding constraints and separating between positive and negative cash flows which need to be remunerated at different interest rates. Brigo and Pallavicini (2014) postulate the existence of a risk-neutral pricing measure, and obtain the valuation equation

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which accounts for counterparty credit risk, funding, and collateral servicing costs. Bielecki and Rutkowski (2014) construct a semimartingale framework and provide a backward stochastic differential equation (BSDE) representation for the wealth process that replicates a default-free claim, assuming the trading parties to be default-free. Building on Bielecki and Rutkowski (2014), Nie and Rutkowski (2018) study the pricing of contracts both from the perspective of the investor and her counterparty, and provide the range of fair bilateral prices. The default risk of the trading parties involved in the transaction is accounted for by Burgard and Kjaer (2011b), who derive the partial differential equation representations for the derivative value, using replication arguments. Andersen et al. (2017) view funding costs from a corporate finance perspective. They develop a model that is consistent with asset pricing theories, and importantly account for the impact of funding strategies on the market valuation of the claim. We refer to Crépey et al. (2014) for an overview of the literature on valuation adjustments.

We consider a market environment, in which an investor transacts credit default swaps with a counterparty and wants to compute the XVA of her trading position. The trading inefficiencies contributing to the XVA include funding costs due to the difference between treasury borrowing and lending rates, losses originating from premature default of the investor or her counterparty, and costs of posting initial and variation margin collateral. The distinguishing feature of our framework, relative to the literature surveyed above, is that the investor is uncertain about the rate of return of the counterparty bond used to hedge counterparty credit risk and hence unable to compute the actual XVA. Several studies have investigated the determinants of bond returns, including default risk and market liquidity. Acharya et al. (2013) bucket the bonds into rating classes, ranging from AAA through CCC. They show that the economic contribution of interest rate and default risks to bond returns is larger than the contribution of liquidity under both stressed and normal market regimes. De Jong and Driessen (2012) show that default risk is the predominant component of returns for investment grade bonds, while liquidity risk plays the most significant role for returns of junk bonds.

We develop a framework of robust pricing for a credit default swap portfolio, that accounts for uncertainty in the return rate of risky bonds. A recent work by Fadina and Schmidt (2018) develops a framework that incorporates model uncertainty into defaultable term structure models. They assume lower and upper bounds for the default intensity and construct uncertainty intervals for the defaultable bond prices, ignoring valuation adjustments. Our theory parallels that for uncertain volatility introduced by Avellaneda et al. (1995). Therein, the authors consider a Black-Scholes type model, in which the volatility of the underlying asset is unknown and only a priori deterministic bounds for its value are prescribed. They derive the Black-Scholes-Barenblatt equation characterizing the value of European options in this model; see also Lyons (1995) for the case of one-dimensional barrier options. Fouque and Ning (2017) generalize the analysis to the case that the volatility fluctuates between two stochastic bounds, arguing that this better captures the behavior of options with longer maturity. Other related works include Hobson (1998), El Karoui et al. (2009), and Denis and Martini (2006) who provide a probabilistic description using the theory of capacities. Another line of subsequent work, building on the uncertain volatility model, is the theory of G-Brownian motion and G-expectation by Peng (2007).

We focus on the impact that uncertainty on the counterparty bond return has on the valuation of the trade, and compute upper and lower bounds for the XVA. There are both similarities and differences between our setup and the uncertain volatility setup of Avellaneda et al. (1995). On the one hand, the differential equations yielding the robust XVA price process are ordinary and of first order, as opposed to the case of uncertain volatility in which the price bounds are obtained by solving
second-order partial differential equations. Additional simplifications arise in our framework because we do not need to deal with the singularity of probability measures. On the other hand, new technical challenges appear due to the complex relationship between the valuation of the replicating portfolio, the determination of collateral levels, and the close-out requirements of the valuation party.

In our framework, the investor uses risky bonds underwritten by herself, her counterparty, and the reference entities, to replicate the XVA process associated with the credit default swaps transaction. There are several reasons behind our choices of using bonds as replicating instruments. First, it is well known (see, for instance, Gregory (2015)) that the vast majority of banks’ counterparties do not have liquidly traded CDSs. A clear example is the municipal bond market that, unlike the corporate bond market, has not sufficiently fostered the development of the corresponding CDS market (Fabozzi and Feldstein (2008)). The bankruptcy events experienced by municipal entities in recent years, such as the city of Detroit, increased pressure on municipal bond investors and exchanged traded funds specializing in municipal bonds to heighten risk management. The most direct instruments to replicate this risk are the single name credit default swaps. However, the market for credit default swaps is still very thin compared to the primary market of municipal bonds. According to Van Deventer (2014), only 11 municipal CDSs have traded since the DTCC began reporting weekly trading activities in July 2010.1 Second, a problem typically encountered when one considers CDS based replication strategies is that each CDS contract will be on-the-run only for the first three months after its issuance. After these three months, the CDS will change its status to off-the-run and become very illiquid. We thus opt for bonds as replicating instruments because they are typically easier to trade at any time prior to maturity than the corresponding off-the-run CDSs.2

We derive the nonlinear valuation equation that uses corporate bonds to replicate the credit default swap position, and take into account counterparty credit risk and closeout payoffs exchanged at default. Our valuation equation is a special BSDE driven by Lévy processes, that contains jump-to-default but no diffusion terms. We characterize the super-replicating price of the transaction as the solution to a nonlinear system of ODEs that is obtained from the nonlinear BSDE tracking the XVA process, after projecting such a BSDE onto the smaller filtration exclusive of credit events information. The system consists of an ODE, whose solution is the value of the transaction cash flows ignoring market inefficiencies, and additional ODEs that yield the XVA of the portfolio. Intuitively, the super-replicating price is the value attributed to the trade by an investor who positions herself in the worst possible economic scenario. We find that the super-replicating price and the corresponding super-replicating strategies may not be recovered by simply plugging one of the extremes of the uncertainty interval into the valuation equation. Our analysis indicates that, depending on the relation between the current value of the XVA replicating portfolio and the close-out value of the transaction, the lower or upper extreme of the uncertainty interval on the counterparty bond rate should be used in the super-replication strategy. The trader wants to be robust against the most negative outcome, and therefore will choose the extreme of the interval that minimizes the instantaneous change in the value of the position. This will in turn require the investor to initially hold the maximal wealth to implement this replicating strategy, hence leading to the maximum initial value of the portfolio. As

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1The Wall Street Journal blog post “Traders Find Short Bets on Puerto Rico a Challenge” claims that default insurance on Puerto Rico, sold in the form of derivatives called credit-default swaps, is available from few dealer banks. The contracts also have barely traded because there is not sufficiently available protection for purchase and disclosures from the Commonwealth have been limited.

2Bielecki et al. (2008) introduce rolling credit default swaps to deal with the illiquidity of off-the-run CDSs. These are non-tradable instruments, that are economically equivalent to a self-financing trading strategy that at any given time $t$ enters into a CDS contract of maturity $T$ and then unwinds the contract at time $t + dt$. 

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long as the portfolio replicates the trade at the terminal time, its initial value provides an upper bound on the value of the XVA. For example, if the strategy replicating the XVA requires, at a given time, the investor to be short the counterparty’s bond, i.e., a positive jump would arise at the counterparty default (this would be the case if the value of the XVA replicating portfolio lies below the close-out value), then the trader would choose to use the upper extreme of the uncertainty interval, because this corresponds to the maximal default intensity and thus minimizes the instantaneous change in value. As the required replicating position may switch from short to long and vice versa several times before the close-out time, the extreme of the default interval used in the valuation of the superreplicating strategy will change, too.

The rest of the paper is organized as follows. We develop the market model in Section 2. We introduce the valuation measure, collateral process and close-out valuation in Section 3. We introduce the replicating wealth process and the notion of arbitrage in Section 4. We develop a robust analysis of the XVA process in Section 5. Section 6 concludes.

2 Model

Our framework builds on that proposed by Bichuch et al. (2017) in that it uses a reduced form model of defaults and maintains the distinction between universal and investor specific instruments. The model economy consists of \( N \) firms, indexed by \( i = 1, \ldots, N \), whose default events constitute the sources of risks in the portfolio. We use \( I \) and \( C \) to denote, respectively, the trader (also referred to as investor throughout the paper) executing the transaction and her counterparty. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space rich enough to support the following constructions. We assume the existence of \( N+2 \) independent and identically distributed unit mean exponential random variables \( E_i; i = 1, \ldots, N, I, C \).

The default time of each firm \( i \) is defined to be the first time its cumulated intensity process exceeds the corresponding exponentially distributed random variable, i.e., \( \tau_i = \sup\{t \geq 0 : \int_{0}^{t} h_i^E(s) ds > E_i\} \).

Accordingly, we use the default indicator process \( H_i(t) = \mathbb{1}_{\{\tau_i \leq t\}}, t \geq 0 \), to track the occurrence of firm \( i \)'s default. The background filtration \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \), where \( \mathcal{F}_t := \sigma(H_j(u); u \leq t; \ j \in \{1, \ldots, N\}) \), contains information about the risk of the portfolio, i.e., of the default of the \( N \) firms referencing the traded securities, but not about the defaults of the investor \( I \) and her counterparty \( C \). The default intensity processes \( (h_i^E(t))_{t \geq 0}, \ i \in \{1, \ldots, N, I, C\} \), are constructed so that they are adapted to the background filtration \( \mathbb{F} \), i.e., the default intensity at a given time \( t \) depends on the firms’ defaults occurring before time \( t \). To achieve this, we use the following stepwise procedure: Assume \( h_i^{E,0}(t) \in \mathcal{F}_0 \otimes \mathcal{B}([0, t]) \) and define \( \tau_i^0 := \sup\{t \geq 0 : \int_{0}^{t} h_i^{E,0}(s) ds > E_i\} \). Then, we can define \( \mathcal{F}_{t}^{1} := \sigma(H_j(u); u \leq t \wedge \tau_i^0; j \in \{1, \ldots, N\}) \), where \( \tau_i^0 \) is the time of the first default (the first order statistics). For \( k \geq 1 \), choose \( f_i^{E,k}(t) \in \mathcal{F}_t^{k} \otimes \mathcal{B}([0, t]) \) and define recursively \( h_i^{E,k}(t) := h_i^{E,k-1}(t) \mathbb{1}_{[0, \tau_i^{k-1}]}(t) + h_i^{E,k}(t) \mathbb{1}_{[\tau_i^{k-1}, \tau_i^{k}]}(t) \), where we use the notation \( \tau_i^{k}(t) \) to denote the \( i \)-th order statistics of the \( k \)-level stopping time \( \tau_i^k \). Then we define \( \tau_i^k := \sup\{t \geq 0 : \int_{0}^{t} h_i^{E,k}(s) ds > E_i\} \) as well as \( \mathcal{F}_{t}^{k+1} := \sigma(H_j(u); u \leq t \wedge \tau_i^{k}; j \in \{1, \ldots, N\}) \) for \( k \in \{1, \ldots, N\} \). In this way, the intensity \( h_i^{E,k} \) agrees with \( h_i^{E,k-1} \) up to the \( k \)-th default, but accounts for information after the \( k \)-th default thereafter. Finally, we define the full filtration \( \mathbb{F} := (\mathcal{F}_{t}^{N+1})_{t \geq 0} \).

We denote the filtration containing information about the investor and counterparty defaults by \( \mathbb{H} := (\mathcal{H}_t)_{t \geq 0} \), where \( \mathcal{H}_t := \sigma(H_j(u); u \leq t; j \in \{I, C\}) \). By construction, the default intensities \( h_i^E, \ i \in \{1, \ldots, N, I, C\} \), are piecewise deterministic functions of time (we thus work in the framework of piecewise-deterministic Markov processes, see Davis (1984)). We furthermore require that they are
piecewise continuous and uniformly bounded. The enlarged filtration, including both portfolio risk (default events of the \(N\) firms referencing portfolio securities) and counterparty risk (default events of investor and her counterparty), is denoted by \(\mathcal{G} = (\mathcal{G}_t)_{t \geq 0} = (\mathcal{F}_t \vee \mathcal{H}_t)_{t \geq 0}\). We will consider the augmented filtrations, i.e., the smallest complete and right-continuous filtrations encompassing the natural filtrations, and denote them by \(\mathcal{F}, \mathcal{H}, \mathcal{G}\) (with a slight abuse of notation). For future purposes, we define the martingale compensator processes \(\varpi_{i,p}\) of \(H_i\) as

\[
\varpi_{i,p}(t) := H_i(t) - \int_0^t (1 - H_i(s)) h_i^p(s) \, ds, \quad i \in \{1, \ldots, N, I, C\}.
\]

By construction, these compensator processes are \(\mathcal{F}\)-martingales for \(i \in \{1, \ldots, N\}\), and \(\mathcal{G}\)-martingales for \(i \in \{1, \ldots, N, I, C\}\).

We take the perspective of a valuation party, responsible for determining the price of the trade and its collateral requirements. Both the valuation party and the trader know the bond rates of the firms in the portfolio as well as the investor bond rate \(\mu_i\). The trader, however, only has limited information about the actual bond rate \(\mu_C = (\mu_C(t))_{t \geq 0}\) of her counterparty, and in particular only knows its upper and lower bound, i.e., \(\underline{\mu}_C \leq \mu_C \leq \overline{\mu}_C\), where \(0 < \underline{\mu}_C \leq -\frac{1}{T} \log(p_C) \leq \overline{\mu}_C < \infty\), where \(p_C\) is the initial price of the bond underwritten by the counterparty.

**Replicating instruments** The goal is for the investor to replicate a portfolio of credit default swaps (CDS) written on \(N\) different reference entities, denoted by \(1, 2, \ldots, N\). All CDSs are assumed to mature at the same time \(T\). The credit risk exposure associated with this portfolio is replicated using both *universal* and *investor specific* instruments. The universal instruments are available to all market participants, while the investor specific instruments are accessible solely to the her and not to other market participants. The universal instruments include (defaultable) bonds underwritten by the reference entities in the credit default swaps portfolio as well as by the trader and her counterparty. Under the physical measure \(\mathbb{P}\), for \(i = 1, \ldots, N, I, C\), and \(0 \leq t \leq T\), the dynamics of the defaultable bond price processes with zero recovery at default are given by

\[
dP_i(t) = \mu_i P_i(t) \, dt - P_i(t^-) \, dH_i(t), \quad P_i(0) = p_i,
\]

where \((\mu_i(t)), i \in \{1, \ldots, N, I, C\}\), are \(\mathcal{F}\)-adapted and thus piecewise deterministic processes, potentially jumping at discrete times corresponding to default events, satisfying \(p_i e^{-\int_0^T \mu_i(s) \, ds} \mathbb{1}_{\{\tau_i > T\}} = 1\) for the initial (observable) bond price \(p_i\). We assume that the rates \(\mu_i, i \in \{1, \ldots, N, I\}\) are observable while the investor has no further information about \(\mu_C\) except for that it is constrained to lie in the interval \([\underline{\mu}_C, \overline{\mu}_C]\).

The *investor specific* instruments include her funding and collateral accounts. We assume that the trader lends and borrows from her treasury desk at, possibly different, rates \(r_f^+\) (the lending rate) and \(r_f^-\) (the borrowing rate). Denote by \(B^r_f\) the cash accounts corresponding to these funding rates. An investment strategy of \(\xi^f := (\xi^f(s); s \geq 0)\) shares in the funding account yields an account value \(B^{r_f} := (B^{r_f}(s); s \geq 0)\) given by

\[
B^{r_f}(t) := B^{r_f}(t)(\xi_f) = e^{\int_0^t r_f(\xi^f(s)) \, ds},
\]

where

\[
r_f := r_f(y) = r_f^+ \mathbb{1}_{\{y < 0\}} + r_f^- \mathbb{1}_{\{y > 0\}}.
\]
Collateral The trader and counterparty use a collateral account to mitigate counterparty risk. Following the standards set by the Basel Committee on Banking Supervision (BCBS) and the International Organization of Securities Commissions (IOSCO) (see BIS Margin (2013)), the collateral consists of variation margins, tracking the changes in market value of the traded portfolio and denoted by $VM$, and initial margins that are used to mitigate the gap risk at the close-out of the transaction and denoted by $IM$. The European Market Infrastructure Regulation (EMIR) posits at least daily updates for variation margins and requires a revaluation of initial margins at least every ten days (see EMIR OTC Regulation (2016)). In the United States, the Commodity Futures Trading Commission requires daily updates on initial margins (CFTC Margin Requirements (2016)). Mathematically, the collateral process $M := (M_t; t \geq 0)$, $M = VM + IM$, is an $\mathbb{F}$ adapted stochastic process which we assume to be positive if the investor posts collateral (is the collateral provider) and negative if she receives collateral (is the collateral taker).

Denote by $r^+_m$ the interest rate on collateral demanded by the investor when she posts to her counterparty, and by $r^-_m$ the rate on collateral demanded by the counterparty when the investor is the collateral taker. The value of the collateral account at time $t$ is then given by

$$B^{rm}(t) = e^{\int_0^t r_m(M_s) ds},$$

where

$$r_m := r_m(x) = r^-_m \mathbb{1}_{\{x<0\}} + r^+_m \mathbb{1}_{\{x>0\}}.$$

Denoting by $\psi^m$ the number of shares of collateral account $B^{rm}_t$ held by the trader at time $t$, we have the following relation

$$\psi^m(t) B^{rm}(t) = -M_t. \tag{4}$$

The collateral amount $M_t$ received or posted at time $t$ will be determined by a valuation party, as discussed in the next section. Figure 1 describes the mechanics of the entire flow of transactions.

3 Valuation Measure, Collateralization and Close-out

We take the perspective of a trader who purchases a portfolio of credit default swaps, and determine its value by constructing a replicating portfolio. Such a portfolio uses bonds of the underlying reference entities to replicate the market risk of the transaction, and bonds of the trading parties to replicate the counterparty risk of the trading parties. Hence, from a corporate perspective, we are interested in the entrance price of the transaction. Because the trader does not know the exact default intensity of her counterparty, such a replication argument can only provide price bounds. In particular, the lower bound provides a reliable benchmark to measure the potential losses incurred if the portfolio is acquired at a higher price.

Remark 3.1. The trader aims to compute the difference between the entrance price, i.e. the price at which the transaction is settled, and the market value of the transaction, so that she can identify the underlying risk factors and allocate them to different desks within the bank. This difference is referred to as XVA.
It is important to introduce a finer distinction between the different sources of surcharges and unreplicable risk (referred to, e.g., as CVA, FVA, KVA) to correctly allocate them to the managing desks. Hence, when calculating the exit price, i.e., the price at which the portfolio can be liquidated on the open market (this is relevant for tax and regulatory purposes), one needs to account for these components at a higher level of granularity. One of these components is the KVA, defined as the amount of capital at risk set aside by shareholders of the investor’s firm. KVA should be calculated under the historical measure, which is typically assumed to be the same as the risk neutral measure to preserve analytical tractability. Our analysis deals with entrance prices, i.e., prior to decomposing the trade into risk sources and splitting it to the various desks. Nevertheless, we compute the (super)replicating price of the transaction, which is robust against the specific choices of physical and pricing measure. Hence, our analysis might also be used to provide bounds for KVA, which are robust to misspecification of the counterparty’s default intensity.

Next, we discuss public and private valuations. Private valuations are based on discount rates, which depend on investor specific characteristics, while public valuations depend on publicly available discount factors. Specifically, public valuations are needed for the determination of collateral requirements and the close-out value of the transaction. They are determined by a valuation agent who might be either one of the parties involved in the transaction or a third party, in accordance with market practices reviewed by the International Swaps and Derivatives Association (ISDA). The valuation agent determines the closeout value of the transaction by calculating the so-called clean price of the derivative, using the discount rate $r_D$ and the bond rates of the firms in the portfolio, $\mu_i$, $i \in \{1, \ldots, N\}$ (we recall that the latter are known to the valuation agent). Throughout the paper, we will use the superscript $\wedge$ when referring specifically to public valuations.

The replicating process will stop before maturity if the trader or her counterparty were to default prematurely. We thus define the terminal time of the trade (i.e., the earliest between the default time of either party or the maturity $T$ of the transaction) as $\tau := \tau_I \wedge \tau_C \wedge T$. The valuation done by the
agent is mathematically represented as pricing the trade under the valuation measure \( Q \) associated with the publicly available discount rate \( r_D \) chosen by the agent. The measure \( Q \) is equivalent to \( \mathbb{P} \) and their relation is specified by the Radon-Nikodým density

\[
\frac{dQ}{dP}\bigg|_{\mathcal{F}_{\tau \wedge (\tau_1 \vee \ldots \vee \tau_N)}} = \prod_{i \in \{1, \ldots, N, I, C\}} \left( \frac{\mu_i - r_D}{h_i^Q} \right)^{H_i(\tau \wedge \tau_i)} \int_0^{\tau \wedge \tau_i} h_i^P(s) ds du.
\]

(5)

**Remark 3.2.** As the valuation measure is used to determine the clean price of the transaction, it needs not depend on the default intensities of the investor \( I \) and her counterparty \( C \). Nevertheless, we have included both of these default intensities in the definition of \( Q \) because this will simplify the exposition in later sections of the paper. In particular, we do not need to introduce a different measure for the investor’s valuation.

The \( Q \)-dynamics of the defaultable bonds can be derived using Girsanov’s theorem and are given by

\[
dP_i(t) = r_D P_i(t) dt - P_i(t-)d\varpi^Q_i(t),
\]

where \( \varpi^Q_i := (\varpi^Q_i(t); 0 \leq t \leq \tau \wedge \tau(N)) \) are \((\mathbb{F}, \mathbb{Q})\)-martingales. They can be represented explicitly as

\[
\varpi^Q_i(t) = \varpi^P_i(t) + \int_0^t (1 - H_i(u))(h_i^P(u) - h_i^Q(u)) du
\]

where the processes \( h_i^Q = \mu_i - r_D, i \in \{1, \ldots, N\} \), (with \( \mu_i, i = 1, \ldots, N, I, C \), being the rate of returns of the bonds underwritten by the reference entities, trader and her counterparty) are the firms’ default intensities under the valuation measure and assumed to be positive.

### 3.1 Collateral

The public valuation process of the credit default swap portfolio, as determined by the valuation agent, is given by

\[
\hat{V}_t = \sum_{i=1}^{N} z_i \hat{C}_i(t),
\]

where \( \hat{C}_i(t) \) is the time \( t \) value of the credit default swap referencing entity \( i \). The variables \( z_i \) indicate if the trader sold the swap to her counterparty \( (z_i = 1) \) or purchased it from her counterparty \( (z_i = -1) \). In the case the swap is purchased, the trader pays the spread times the notional to her counterparty, and receives the loss rate times the notional at the default time of the reference entity, if it occurs before the maturity \( T \). This is the so-called “clean price”, and does not account for credit risk of the counterparty, collateral or funding costs. Clearly, the public valuation of the portfolio is just the sum of the valuation of the individual CDSs.

Consistently with market practices, the collateralization process consists of two parts, the initial margin and the variation margin. The variation margin is set to be a fixed ratio of the public valuation of the portfolio, while the initial margin is designed to mitigate the gap risk and is calculated using value at risk. Such a risk measure is set to cover a number of days of adverse price/credit spread movements for the portfolio position with a target confidence level.\(^4\) Note that there is an important difference between initial and variation margins. Variation margins are always directional and can be rehypotecated (i.e., it flows from the paying party to the receiving party; the latter may use it for

\(^4\)Both EU and US authorities require initial margins to cover losses over a liquidation period for ten days in 99% of all realized scenarios (EMIR OTC Regulation (2016); CFTC Margin Requirements (2016)).
investment purposes), whereas initial margins have to be posted by both parties and need to be kept in a segregated account, thus they cannot be used for portfolio replication. Moreover, we assume that collateral is posted and received in the form of cash, which is practically the most common form of collateral.\footnote{More precisely, cash is the predominant form of collateral used for variation margins, and it accounts for about 80\% of the total posted variation margin amount. Initial margins are usually delivered in the form of government securities (see, for instance, page 7 of ISDA (2017)). Overall, the amount of variation margin posted for bilaterally cleared derivatives contracts was about $173 billion in 2017, whereas the variation margin accounted for $870 billions (see page 1 therein).}

Thus, on the event that neither the trader nor her counterparty have defaulted by time $t$, and the reference entities in the portfolio have not all defaulted, the collateral process is defined as

$$M_t := IM_t + VM_t = \left(\beta \left(VaR_q(\hat{V}_{(t+\delta)\wedge T} - \hat{V}_t \mid \tau \wedge \tau(N) > t)\right)^+ + \alpha \hat{V}_t\right) \mathbb{1}_{\{\tau \wedge \tau(N) > t\}}, \tag{6}$$

where $0 \leq \alpha \leq 1$ is the collateralization level, $\delta > 0$ is the delay in collateral posting, $q$ is the level of risk tolerance and $\beta$ is stress factor. The case $\alpha = 0$ corresponds to zero collateralization, while $\alpha = 1$ means that the transaction is fully collateralized. The positive part of the value at risk quantity captures the fact that initial margins cannot be re-hypothecated. Hence, the wealth process associated with the investor’s trading strategy does not include received initial margins.

### 3.2 Close-out value of transaction

We follow the risk-free closeout convention in the case of default by the trader or her counterparty. According to this convention, each party liquidates the position at the market value when the other trading counterparty defaults. Hence, the value of the replicating portfolio will coincide with the third party valuation if the amount of available collateral is sufficient to absorb all occurred losses. If this is not the case, the trader will only receive a recovery fraction of her residual position, i.e., after netting losses with the available collateral. Let us denote by $\theta$ the value of the replicating portfolio at $\tau < T$. This is given by

$$\theta := \hat{V}_\tau + \mathbb{1}_{\{\tau_C < \tau_I\}} L_C Y^- - \mathbb{1}_{\{\tau_I < \tau_C\}} L_I Y^+$$

where $Y := \hat{V}_\tau - M_{\tau -} = (1 - \alpha)\hat{V}_\tau - \beta \left(VaR_q(\hat{V}_{(\tau+\delta)\wedge T} - \hat{V}_\tau)\right)^+$ is the value of the claim at default netted of the posted collateral, and 0 $\leq L_I, L_C \leq 1$ are the loss rates on the trader and counterparty claims, respectively. Alternatively, we can represent the value of the portfolio at default as

$$\theta = \theta(\tau, \hat{V}, M) = \mathbb{1}_{\{\tau_I < \tau_C\}} \theta_I(\hat{V}_\tau, M_{\tau -}) + \mathbb{1}_{\{\tau_C < \tau_I\}} \theta_C(\hat{V}_\tau, M_{\tau -}),$$

where we define

$$\theta_I(\hat{v}, m) := \hat{v} - L_I(\hat{v} - m)^+ \quad \text{and} \quad \theta_C(\hat{v}, m) := \hat{v} + L_C(\hat{v} - m)^-$$

and recall

$$M_t = \alpha \hat{V}_t + \beta \left(VaR_q(\hat{V}_{(t+\delta)\wedge T} - \hat{V}_t \mid \tau \wedge \tau(N) > t)\right)^+.$$
4 Wealth process & Arbitrage

We consider a stylized model of single name credit default swaps, and view all exchanged cash flows from the point of view of the protection seller. If the trader purchases protection from her counterparty against the default of the \(i\)-th firm, then the trader makes a stream of continuous payments at a rate \(S_i\) of the notional to her counterparty, up until contract maturity or the arrival of the credit event, whichever occurs earlier. Upon arrival of the default event of the \(i\)-th firm, and if this occurs before the maturity \(T\), the protection seller pays to the protection buyer the loss on the notional, obtained by multiplying the loss rate \(L_i\) by the notional. As the notional enters linearly in all calculations, we fix it to be one.

Recall that \(\xi_i\) denotes the number of shares of the bond underwritten by the reference entity \(i\), \(\xi_f\) the number of shares in the funding account, and we use \(\xi_I\) and \(\xi_C\) to denote the number of shares of trader and counterparty bonds, respectively. Using the identity (4), we may write the wealth process as a sum of contributions from each individual account:

\[
V_t := \sum_{i=1}^N \xi_i^t P_i^t + \xi_I^t P_I^t + \xi_C^t P_C^t + \xi_f^t B_r^t - \psi_m^t B_r^m.
\] (7)

For the purpose of arbitrage-free valuation, it is important to consider not only the actual CDS portfolio, but an arbitrary multiple of it. Hence, we will consider a multiple \(\gamma\) of the acquired portfolio, and focus on self-financing strategies.

**Definition 4.1.** A collateralized trading strategy \(\phi := (\xi_1^t, \ldots, \xi_N^t, \xi_I^t, \xi_C^t, \xi_f^t, t \geq 0)\) associated with \(\gamma\) shares of a portfolio \(w = (w_1, w_2, \ldots, w_N)\), where \(w_i \in \{0, 1\}\) for \(i = 1, \ldots, N\), is self-financing if, for \(t \in [0, \tau \wedge \tau(N)]\), it holds that

\[
V_t(\gamma) := V_0(\gamma) + \sum_{i=1}^N \int_0^t \xi_i^u dP_i^u + \int_0^t \xi_I^u dP_I^u + \int_0^t \xi_C^u dP_C^u + \int_0^t \xi_f^u dB_r^u - \int_0^t \psi_m^u dB_r^m + \gamma \sum_{i=1}^N w_i S_i (\tau_i \wedge t).
\] (8)

The above expression takes into account the spread payments received/paid by the investor for all CDS contracts which she sold to (resp. purchased from) her counterparty. The set of admissible trading strategies consists of \(\mathbb{F}\)-predictable processes \(\phi\) such that the portfolio process \(V_t(\gamma)\) is bounded from below (cf. Delbaen and Schachermayer (2006)).

**Remark 4.2.** Note that the spread payments received by the trader are continuously reinvested into the replicating instruments (risky bonds) or deposited in the funding account. This is a different setup then that in Bielecki et al. (2008) where the spread payments are used to increase the positions in the CDS instruments. The difference stems from the fact that, in their model, the CDS contracts are liquidly traded replicating instruments, whereas in our case they are part of the portfolio to be replicated.

Before discussing the arbitrage-free valuation of the CDS portfolio, we have to clarify the assumptions under which the underlying market is free of arbitrage from the investor’s perspective (conceptually, we follow (Bielecki and Rutkowski, 2014, Section 3)). Thus, to start with, we exclude the CDS instruments from our consideration, and consider a trader who is only allowed to buy or sell defaultable bonds (written on the reference entities, her counterparty or the investor’s firm itself) and to borrow or lend money from the treasury desk.
Definition 4.3. The market \((P_1, P_2, \ldots, P_N, P^I, P^C)\) admits investor’s arbitrage if, given a non-negative initial capital \(x \geq 0\), there exists an admissible trading strategy \(\varphi = (\xi^1, \xi^2, \ldots, \xi^N, \xi^I, \xi^C)\) such that \(\mathbb{P}[V_T \geq e^{r_I^T r} x] = 1\) and \(\mathbb{P}[V_T > e^{r_I^T r} x] > 0\). If the market does not admit investor’s arbitrage for a given level \(x \geq 0\) of initial capital, the market is said to be arbitrage free from the investor’s perspective.

We impose the following assumption and argue that it provides a necessary and sufficient condition for the absence of arbitrage.

Assumption 4.4. \(r^D \vee r^I_f < \min_{i \in \{1, \ldots, N, I, C\}} \mu_i \wedge \mu_C\).

Remark 4.5. Necessity: The condition \(r^D < \min_{i \in \{1, \ldots, N, I, C\}} \mu_i\) is needed for the existence of the valuation measure defined in Eq. (5) \((h^Q_i = \mu_i - r^D\) and risk-neutral default intensities must be positive). Thus, we should impose \(\mu_C > r^D\), but as the true bond rate \(\mu_C\) is unobservable, we impose instead the slightly stronger \(\mu_C > r^D\). The condition \(r^I_f < \min_{i \in \{1, \ldots, N, I\}} \mu_i \wedge \mu_C\) has an even more practical interpretation because it precludes the arbitrage opportunity of short selling the risky bonds while investing the proceeds in the funding account. Strictly speaking, the condition \(r^D < \mu_I \wedge \mu_C\) is not necessary from an arbitrage point of view, because it addresses only the soundness of the market from the perspective of the valuation party. While \(r^D < \min_{i \in \{1, \ldots, N\}} \mu_i\) is necessary to conclude that the valuation party’s market model is free of arbitrage, one might hypothesize a situation in which \(r^D \geq \mu_I \wedge \mu_C\). From a practical perspective, this is however rather unlikely, as \(r^D\) is typically assumed to be an overnight index swap (OIS) rate and as such lower than the return rates of the defaultable bonds.

Having argued about the necessity in the above remark, we show that Assumption 4.4 is also sufficient to guarantee that the underlying market (i.e., excluding the credit default swap securities) is free of arbitrage. The proof proceeds along very similar lines as Proposition 4.4 in Bichuch et al. (2017), and is delegated to the appendix.

Proposition 4.6. Under Assumption 4.4, the model does not admit arbitrage opportunities for the investor for any \(x \geq 0\).

As in Bichuch et al. (2017), we will define the notion of an arbitrage free price of a derivative security from the investor’s perspective. We assume that the investor has zero initial capital, or equivalently, she does not have liquid initial capital that can be used for replicating the claim until maturity. The replicating portfolio will thus be entirely financed by purchases/sales of the risky bonds via the funding account.

Definition 4.7. The valuation \(P \in \mathbb{R}\) of a derivative security with terminal payoff \(\vartheta \in \mathcal{F}_T\) is called investor’s arbitrage-free if for all \(\gamma \in \mathbb{R}\), buying \(\gamma\) securities for \(\gamma P\) and trading in the market with an admissible strategy and zero initial capital, does not create investor’s arbitrage.

Let \(V_t\) represent the price process of the replicating portfolio, and given by the supremum over all arbitrage free prices. Then we define the total valuation adjustment \(\text{XVA}\) as the difference between this upper arbitrage price and the clean price, i.e.,

\[
\text{XVA}_t(\gamma) = V_t(\gamma) - \gamma \hat{V}_t.
\]

\(\text{XVA}\) thus quantifies the total costs (including collateral, funding, and counterparty risk related costs) incurred by the trader to replicate the sold CDS portfolio. Notice that, at time \(t\), the investor does...
not know the actual counterparty bond rate $\mu_C$ for the time interval $[t, \tau]$. Hence, she is not able to execute the replication strategy yielding the value process $V$, because all what she knows about the bond rate is that $\mu_C \leq \mu_C \leq \bar{\mu}_C$. Therefore, she will have to consider the worst case, accounting for all possible $\mathbb{F}$-predictable dynamics of the bond rate process in the interval $[\mu_C, \bar{\mu}_C]$. Denote the valuation of the replicating portfolio when $\mu_C = \mu$ by $V^\mu$. The robust XVA is defined as

$$r\text{XVA}_t(\gamma) = \text{ess sup}_{\mu \in [\mu_C, \bar{\mu}_C]} V_t^\mu(\gamma) 1_{t<\tau} + V_t(\gamma) 1_{t\geq \tau} - \gamma \hat{V}_t. \tag{10}$$

Notice that the supremum is taken over all admissible valuations only prior to the occurrence of the trader or her counterparty's default. In particular, the valuation process at and after default depends only on the closeout value and thus does not depend on the extremes $\mu_C$ and $\bar{\mu}_C$ of the uncertainty interval.

## 5 Robust XVA for Credit Swaps

In this section, we derive explicit representations for the robust XVA of a credit default swap portfolio. To highlight the main mathematical arguments and economic implications of the results, we start analyzing the case of a single credit default swap in Section 5.1. We develop a comparison argument to establish the uniqueness of the robust XVA process and of the corresponding super-replicating strategies in Section 5.2. We provide an explicit computation of margins under the proposed framework in Section 5.3. We generalize the analysis to a portfolio of credit default swaps in Section 5.4.

### 5.1 BSDE representation of XVA

This section characterizes the XVA process given in Eq. (9) as the solution to a BSDE. We start analyzing the dynamics of the process $V_t(\gamma)$. Given a self financing strategy, the investor’s wealth process in (8) under the risk neutral measure $\mathbb{Q}$ follows the dynamics

$$dV_t(\gamma) = (r_f \xi_t^1 B_t^{r_f} + r_D \xi_t^1 P_t^1 + r_D \xi_t^C P_t^C - r_m \psi_t^m B_t^r + \gamma S_t) \, dt$$

$$- \xi_t^1 P_t^1 \, d\mathcal{W}_t^Q - \xi_t^1 P_{t-} \, d\mathcal{W}_t^{C,Q} - \xi_t^C P_t^C \, d\mathcal{W}_t^{C,Q}$$

$$= \left( r_f^+ (\xi_t^1 B_t^{r_f})^+ - r_f^- (\xi_t^1 B_t^{r_f})^- + r_D \xi_t^1 P_t^1 + r_D \xi_t^C P_t^C \right) \, dt$$

$$+ \left( r_m^+ (M_t)^+ - r_m^- (M_t)^- + \gamma S_t \right) \, dt - \xi_t^1 P_{t-} \, d\mathcal{W}_t^Q - \xi_t^1 P_{t-} \, d\mathcal{W}_t^{C,Q} - \xi_t^C P_t^C \, d\mathcal{W}_t^{C,Q}. \tag{11}$$

Setting

$$Z_t^{1,\gamma} := -\xi_t^1 P_{t-}, \quad Z_t^{1,\gamma} := -\xi_t^1 P_{t-}, \quad Z_t^{C,\gamma} := -\xi_t^C P_{t-}, \quad \tag{12}$$

and using Eq. (7), we obtain that

$$\xi_t^1 B_t^{r_f} = V_t(\gamma) - \xi_t^1 P_t^1 - \xi_t^C P_t^C - M_t. \tag{13}$$

We may then rewrite the wealth dynamics as

$$dV_t(\gamma) = \left( r_f^+ (V_t(\gamma) + Z_t^{1,\gamma} + Z_t^{1,\gamma} + Z_t^{C,\gamma} - |\gamma| M_t)^+ - r_f^- (V_t(\gamma) + Z_t^{1,\gamma} + Z_t^{1,\gamma} + Z_t^{C,\gamma} - |\gamma| M_t)^- \right.$$

$$- r_D Z_t^{1,\gamma} - r_D Z_t^{1,\gamma} - r_D Z_t^{C,\gamma} + r_m^+ |\gamma| M_t^+ - r_m^- |\gamma| M_t^- + \gamma S_t) \, dt$$

$$+ Z_t^{1,\gamma} \, d\mathcal{W}_t^Q + Z_t^{1,\gamma} \, d\mathcal{W}_t^{C,Q} + Z_t^{C,\gamma} \, d\mathcal{W}_t^{C,Q}. \tag{14}$$
To study the robust replicating strategy, we use the above dynamics to formulate the BSDE associated with the portfolio replicating the credit default swap. This is given by

\[-dV_t(\gamma) = f(t, V_t(\gamma), Z_t^{1,\gamma}, Z_t^{I,\gamma}, Z_t^{C,\gamma}, \gamma; M_t) \, dt - Z_t^{1,\gamma} \, d\xi_t^{1,Q} - Z_t^{I,\gamma} \, d\xi_t^{I,Q} - Z_t^{C,\gamma} \, d\xi_t^{C,Q},\]

\[V_{\tau \wedge \tau_1}(\gamma) = \gamma L_1 \mathbb{1}_{\tau_1 < \tau} + \theta_I(\gamma \check{V}_\tau, |\gamma| M_{\tau-}) \mathbb{1}_{\{\tau < \tau_1 \wedge \tau \wedge T\}} + \theta_C(\gamma \check{V}_\tau, |\gamma| M_{\tau-}) \mathbb{1}_{\{\tau < \tau_1 \wedge \tau \wedge T\}}, \quad (15)\]

where the driver \( f : \Omega \times [0, T] \times \mathbb{R}^5, (\omega, t, v, z, z^I, z^C, \gamma) \mapsto f(t, v, z, z^I, z^C, \gamma; M) \) is given by

\[f(t, v, z^1, z^I, z^C, \gamma; M) := -(r_f^+(v + z^1 + z^I + z^C - |\gamma| M_t) + r_f^-(v + z^1 + z^I + z^C - |\gamma| M_t) - r_D z^1 - r_D z^I - r_D z^C + r_m^+ |\gamma| M_t^+ - r_m^- |\gamma| M_t^- + \gamma S_1). \quad (16)\]

In the above expression, we highlight the dependence on the collateral process \( M \) that is used to mitigate the default losses associated with the \( \gamma \) units of the traded CDS contract. In the case the reference entity defaults before the investor or her counterparty, \( \tau_1 < \tau \), the terminal condition is given by the loss term \(-\gamma L_1\). This is consistent with the fact that, at this time, the value of the transaction from the investor’s point of view corresponds with the third party valuation \( \gamma \hat{V}_{\tau_1} = \hat{C}_1(\tau_1) = L_1 \mathbb{1}_{\{\tau_1 \leq T\}} \). By positive homogeneity of the driver \( f \) with respect to \( \gamma > 0 \), we will assume that \( \gamma = 1 \) throughout the paper and suppress it from the superscript. The case \( \gamma = -1 \) follows from symmetric arguments.

Next, we study the dynamics of the credit default swap price process \( \hat{V} \), viewed from the valuation agent’s perspective. Such a process satisfies a BSDE that can be derived similarly to Eq. (15) (essentially ignoring the terms \( Z^I, Z^C \) as well as the collateral terms, setting \( r_f^- = r_f^+ = r_D \), and normalizing \( \gamma = 1 \)). This is given by

\[-d\hat{V}_t = (-r_D \hat{V}_t - S_1) \, dt - \hat{Z}_t^I \, d\xi_t^{I,Q},\]

\[\hat{V}_{\tau_1 \wedge T} = L_1 \mathbb{1}_{\tau_1 < T}. \quad (17)\]

This BSDE is well known to admit the unique solution \((\hat{V}_t, \hat{Z}_t^I)\), where \( \hat{V} \) can be represented explicitly (see the Appendix) as

\[\hat{V}_t = \hat{C}_1(t) = -\mathbb{E}^Q \left[ \int_t^T e^{-\int_t^u h_1^Q(s) + r_D ds} S_1 \, du - \int_t^T L_1 h_1^Q(u) e^{-\int_t^u h_1^Q(s) + r_D ds} \, du \right] \mathbb{1}_{\{t \leq \tau_1\}}. \quad (18)\]

We immediately obtain a BSDE for the XVA process given by

\[-dXVA_t = \tilde{f}(t, XVA_t, \tilde{Z}_t^I, \tilde{Z}_t^I, \tilde{Z}_t^C; M) \, dt - \tilde{Z}_t^I \, d\xi_t^{I,Q} - \tilde{Z}_t^I \, d\xi_t^{C,Q},\]

\[XVA_{\tau \wedge \tau_1} = \tilde{\theta}_C(\hat{V}_\tau, M_{\tau-}) \mathbb{1}_{\{\tau < \tau_1 \wedge \tau \wedge T\}} + \tilde{\theta}_I(\hat{V}_\tau, M_{\tau-}) \mathbb{1}_{\{\tau < \tau_1 \wedge \tau \wedge T\}}, \quad (19)\]

where

\[\tilde{Z}_t^I := Z_t^I - \hat{Z}_t^I, \quad \tilde{Z}_t^I := Z_t^I, \quad \tilde{Z}_t^C := Z_t^C,\]

\[\tilde{\theta}_C(\hat{v}, m) := L_C(\hat{v} - m)^-, \quad \tilde{\theta}_I(\hat{v}, m) := -L_I(\hat{v} - m)^+, \quad (20)\]

and

\[\tilde{f}(t, xva, \tilde{Z}_t^I, \tilde{Z}_t^C; M) := -(r_f^+(xva + \tilde{Z}_t^I + \tilde{Z}_t^C + L_1 - M_t) + r_f^-(xva + \tilde{Z}_t^I + \tilde{Z}_t^C + L_1 - M_t) - r_D \tilde{Z}_t^I - r_D \tilde{Z}_t^C + r_m^+ (M_t)^+ - r_m^- (M_t)^- - r_D L_1). \quad (21)\]
Above, we have used the fact that $\tilde{Z}_1^l = L_1 - \tilde{V}_t = L_1 - \tilde{V}_t$ by stochastic continuity and thus $Z_1^l = \tilde{Z}_1^l + \tilde{Z}_1^l = \tilde{Z}_1^l + L_1 - \tilde{V}_t$.

We can now apply the reduction technique developed by Crépey and Song (2015) to find a continuous ordinary differential equation describing the XVA prior to the investor and her counterparty’s default.

**Proposition 5.1.** The BSDE

$$-d\tilde{U}_t = \tilde{g}(t, \tilde{U}_t; \tilde{V}, M) \, dt,$$

$$\tilde{U}_T = 0,$$

in the (trivial) filtration $\mathbb{F}$, with driver

$$\tilde{g}(t, \tilde{u}; \tilde{V}, M) = h^Q_I(\tilde{\theta}_I(\tilde{V}_t, M_t) - \tilde{u}) + h^Q_C(\tilde{\theta}_C(\tilde{V}_t, M_t) - \tilde{u}) - h^Q_I \tilde{u}$$

$$+ \int (t, \tilde{u}, \tilde{u}_{\tilde{V}_t, M_t} - \tilde{u}, \tilde{\theta}_C(\tilde{V}_t, M_t) - \tilde{u}; M)$$

admits a unique solution $\tilde{U}$, that is related to the unique solution $(XVA, \tilde{Z}^l, \tilde{Z}^l, \tilde{Z}^C)$ of the BSDE in Eq. (19) as follows. On the one hand

$$\tilde{U}_t := XVA_{t \land (\tau \wedge \tau_1)}$$

is a solution to the ODE (reduced BSDE) in Eq. (22), and on the other hand a solution to the full XVA BSDE (19) is given by

$$XVA_t = \tilde{U}_t \mathbb{I}_{\{t < \tau \wedge \tau_1\}} + \left( \tilde{\theta}_C(\tilde{V}_{\tau_1}, M_{\tau_1}) \mathbb{I}_{\{\tau_1 < \tau \wedge \tau_1 \wedge T\}} + \tilde{\theta}_I(\tilde{V}_t, M_t) \mathbb{I}_{\{\tau_1 < \tau \wedge \tau_1 \wedge T\}} \right) \mathbb{I}_{\{t \geq \tau \wedge \tau_1\}},$$

$$\tilde{Z}_t^l = -\tilde{U}_t \mathbb{I}_{\{t \leq \tau \wedge \tau_1\}},$$

$$\tilde{Z}_t^l = \left( \tilde{\theta}_I(\tilde{V}_t, M_t) - \tilde{U}_t \right) \mathbb{I}_{\{t \leq \tau \wedge \tau_1\}},$$

$$\tilde{Z}_t^C = \left( \tilde{\theta}_C(\tilde{V}_t, M_t) - \tilde{U}_t \right) \mathbb{I}_{\{t \leq \tau \wedge \tau_1\}}.$$

The uniqueness of the solution to the original BSDE for $V$ as well as to their projected versions in the $\mathbb{F}$-filtration follows from the definition of XVA.

**Corollary 5.2.** The BSDE (15) admits a unique solution. This solution is related to the unique solution $\tilde{U}$ of the ODE

$$-d\tilde{U}_t = g(t, \tilde{U}_t; \tilde{V}, M) \, dt,$$

$$\tilde{U}_T = 0,$$

in the filtration $\mathbb{F}$ with

$$g(t, \tilde{u}; \tilde{V}, M) = h^Q_I(\theta_I(\tilde{V}_t, M_t) - \tilde{u}) + h^Q_C(\theta_C(\tilde{V}_t, M_t) - \tilde{u}) - h^Q_I \tilde{u}$$

$$+ \int (t, \tilde{u}, L_1 - \tilde{u}, \theta_I(\tilde{V}_t, M_t) - \tilde{u}, \theta_C(\tilde{V}_t, M_t) - \tilde{u}; M)$$

via the following relations. On the one hand

$$\tilde{U}_t := V_{t \land (\tau \wedge \tau_1)}$$

is a solution to the reduced BSDE (26), while on the other hand a solution to the full BSDE (15) is given by

$$V_t := \tilde{U}_t \mathbb{I}_{\{t < \tau \wedge \tau_1\}} + \left( L_1 \mathbb{I}_{\tau_1 < \tau} + \theta_C(\tilde{V}_{\tau_1}, M_{\tau_1}) \mathbb{I}_{\{\tau_1 < \tau \wedge \tau_1 \wedge T\}} + \theta_I(\tilde{V}_t, M_t) \mathbb{I}_{\{\tau_1 < \tau \wedge \tau_1 \wedge T\}} \right) \mathbb{I}_{\{t \geq \tau \wedge \tau_1\}},$$

$$Z_1^l := L_1 - \tilde{U}_t \mathbb{I}_{\{t < \tau \wedge \tau_1\}},$$

$$Z_1^l := \left( \theta_I(\tilde{V}_t, M_t) - \tilde{U}_t \right) \mathbb{I}_{\{t \leq \tau \wedge \tau_1\}},$$

$$Z_1^C := \left( \theta_C(\tilde{V}_t, M_t) - \tilde{U}_t \right) \mathbb{I}_{\{t \leq \tau \wedge \tau_1\}}.$$
Using the above representation, we can provide explicit representations for the replication strategies of the XVA. We will use the tilde symbol (\(\tilde{\cdot}\)) to denote these replicating strategies (e.g., \(\tilde{\xi}^l, \tilde{\xi}^t, \tilde{\xi}^C\) denote, respectively, the number of shares of the bond underwritten by the reference entity, trader and her counterparty) so to distinguish them from the strategies used to replicate the CDS price process. Using the martingale representation theorem for the probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\) and bond price dynamics, we obtain that

\[
\tilde{\xi}_t^l = -\frac{\tilde{Z}_t^l}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}} = \frac{\tilde{U}_t}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}}.
\]  

(27)

Invoking Theorem 5.1 along with equations (12) and (20) we conclude that

\[
\begin{align*}
\tilde{\xi}_t^l &= -\frac{\tilde{Z}_t^l}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}} = \frac{L_t(\tilde{V}_t - M_t^-)^+ + \tilde{U}_t}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}}, \\
\tilde{\xi}_t^C &= -\frac{\tilde{Z}_t^C}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}} = \frac{-L_C(\tilde{V}_t - M_t^-)^- + \tilde{U}_t}{P_t^-} \mathbb{1}_{\{t < \tau \land \tau_1\}},
\end{align*}
\]

(28)\(\) and (29)

and from equations (4) and (6) it follows that

\[
\tilde{\psi}_t^m = \frac{M_t^-}{B_t^m} \mathbb{1}_{\{\tau \land \tau_1 > t\}}.
\]  

(30)

Finally, using Eq. (13) and the identity \(V_t = XVA_t + \tilde{V}_t\), we obtain

\[
\begin{align*}
\tilde{\xi}_t^l &= \frac{V_t - \tilde{V}_t - \tilde{\xi}_t^l P_t^- - \tilde{\xi}_t^C P_t^- - M_t^-}{B_t^-} \mathbb{1}_{\{\tau \land \tau_1 > t\}} \\
&= -\frac{2\tilde{U}_t + L_C(\tilde{V}_t - M_t^-)^- - L_l(\tilde{V}_t - M_t^-)^+ - M_t^-}{B_t^-} \mathbb{1}_{\{\tau \land \tau_1 > t\}},
\end{align*}
\]

(31)

where in the last equality above, we have used the definition of XVA given in Eq. (9) together with the identity \(XVA_t = \tilde{U}_t\) on \(\{\tau \land \tau_1 > t\}\).

Note that the replicating strategies are specified only in terms of bond prices and are thus known to the investor at time \(t\). However, they neither give information on the value of the XVA process nor on the evolution of the replicating strategy, because the default intensity process \(\tilde{h}_C^Q\) is unknown to the investor.

### 5.2 Comparison Pricing and Super-replicating Strategies

This section develops a comparison principle for the reduced BSDE (22) solves by the XVA process. We subsequently use this result to construct a super-replicating strategy for the XVA.

The BSDE given in Eq. (22) is effectively an ODE. To maintain consistency with the theory of ODEs, we switch the direction of time by defining \(\hat{\nu}(t) := \tilde{V}_{T^-} \) and \(m(t) := M_{T^-}\). It follows from Eq. (18) that \(\hat{\nu}\) is bounded, i.e., \(|\hat{\nu}| \leq M_0\) for some constant \(M_0\). Similarly, set \(\hat{u}(t) = \tilde{U}_{T^-}\). Applying the reduction technique of Crépey and Song (2015) to Eq. (17), similarly to how it was done above in Proposition 5.1, we get

\[
\begin{align*}
\partial_t \hat{\nu} &= -(S_1 - \hat{h}_1^Q L_1) - (\hat{h}_1^Q + r_D)\hat{\nu}, \\
\hat{\nu}(0) &= 0.
\end{align*}
\]

(32)
We may then rewrite Eq. (22) as
\[ \begin{align*}
\partial_t \tilde{u} &= \tilde{g}(t, \tilde{u}; \hat{v}, m), \\
\tilde{u}(0) &= 0.
\end{align*} \tag{33} \]

Taken together, the two ODES (32) and (33) constitute a system of ODEs. The functions \( h_t^Q(t), h_t^Q(t), h_C^Q(t) \), \( t \in [0, T] \), are all piecewise (deterministic) continuous. The following theorem, whose proof is reported in the appendix, provides an existence and uniqueness result.

**Proposition 5.3.** There exists a unique (piecewise) classical solution to the system of ODEs (32)-(33).

The following comparison principle theorem, whose proof is reported in the appendix, will be used to find the super-replicating price of the XVA.

**Theorem 5.4** (Comparison Theorem). Assume that there exists \( \bar{\mu}_C \geq \mu_C > r_D \) such that \( \bar{\mu}_C \geq \mu_C^Q \geq \mu_C^\ast \) and let \( \tilde{u} \) be the solution of ODE (33). Let
\[ \begin{align*}
(\mu_C)^\ast(\hat{v}, m, \hat{\tilde{u}}) &= \bar{\mu}_C [\bar{\theta}_C(\hat{v}, m) - \bar{\tilde{u}}] + \mu_C^Q [\bar{\theta}_C(\hat{v}, m) - \bar{\tilde{u}}], \\
(\mu_C)_*(\hat{v}, m, \hat{\tilde{u}}) &= \bar{\mu}_C [\bar{\theta}_C(\hat{v}, m) - \bar{\tilde{u}}] + \mu_C^Q [\bar{\theta}_C(\hat{v}, m) - \bar{\tilde{u}}],
\end{align*} \]
and define the drivers \( g^\ast \) and \( g_\ast \) by plugging the default intensities \( (\mu_C)^\ast \) and \( (\mu_C)_\ast \) into the expression of \( \tilde{g} \) given by (23), i.e.,
\[ \begin{align*}
g^\ast(t, \hat{\tilde{u}}; \hat{v}, m) &= h_t^Q(\hat{\theta}_t(\hat{v}, m_t) - \hat{\tilde{u}}) + ((\mu_C)^\ast(\hat{v}, m_t, \hat{\tilde{u}}) - r_D)(\bar{\theta}_C(\hat{v}, m_t) - \hat{\tilde{u}}) - h_t^Q \hat{\tilde{u}} \\
&+ \hat{f}(t, \hat{\tilde{u}}, \hat{v}, m) \\
g_\ast(t, \hat{\tilde{u}}; \hat{v}, m) &= h_t^Q(\bar{\theta}_t(\hat{v}, m_t) - \hat{\tilde{u}}) + ((\mu_C)_*(\hat{v}, m_t, \hat{\tilde{u}}) - r_D)(\bar{\theta}_C(\hat{v}, m_t) - \hat{\tilde{u}}) - h_t^Q \hat{\tilde{u}} \\
&+ \hat{f}(t, \hat{\tilde{u}}, \hat{v}, m).
\end{align*} \]

Let \( \tilde{u}^\ast \) and \( \hat{u}_\ast \) be the solutions to ODE (33) where \( \tilde{g} \) is replaced by \( g^\ast \) and \( g_\ast \) respectively, i.e.,
\[ \begin{align*}
\partial_t \tilde{u}^\ast &= g^\ast(t, \tilde{u}^\ast; \hat{v}, m), & \tilde{u}^\ast(0) &= 0, \\
\partial_t \hat{u}_\ast &= g_\ast(t, \hat{u}_\ast; \hat{v}, m), & \hat{u}_\ast(0) &= 0.
\end{align*} \tag{34} \]

Then \( \tilde{u}^\ast \leq \hat{u} \leq \tilde{u}^\ast \).

The valuation process calculated based on the extremes of the uncertainty interval \( (\mu_C)^\ast \) and \( (\mu_C)^\ast \) are denoted, respectively, by \( V^{(\mu_C)^\ast} \) and \( V^{(\mu_C)^\ast} \). respectively.

The ODEs (34) may be understood as the credit risk counterparts of the Black-Scholes-Barenblatt PDEs for the uncertain volatility model; see Avellaneda et al. (1995). The main difference between our study and theirs is that, in their paper, the uncertainty comes from the volatility which appears as a second order term in the differential operator. Hence, the indicator function specifying the value of volatility to use in the pricing formula depends on the second order derivative of the option price with respect to the underlying, i.e., the Gamma of the option. In our setting, the indicator function specifying the value of counterparty’s default intensity to use depends on the relation between the current value of the XVA replication and the close-out value. The value of the replicating trade jumps to the close-out value when the counterparty defaults. If the size of this jump is positive, i.e., the close-out value of the transaction is higher, then the trader needs to be short the counterparty’s bond.
to replicate this jump-to-default risk. As the trader wants to consider the worst possible scenario for her trade, she would choose the largest value of the counterparty bond rate $\bar{p}_C$ because this yields the lowest rate of return on her short position. Vice-versa, if the jump is negative, the trader needs to be long the counterparty bond. Consequently, the trader would use the smaller counterparty’s default intensity $h_C$ to account for the worst possible replication scenario.

Our objective is to provide a tight upper bound for the XVA price process, because this would imply a tight super-replicating price. We achieve this by connecting such a super-replicating price to the rXVA defined in Eq. (10). Define the process $\bar{U}_t := \tilde{u}^*(T - t)$. The following theorem shows that the rXVA coincides with the super-replicating price, and additionally specifies the super-replicating strategy. The latter is obtained by taking the strategy given in (27)-(31) and using the super-replicating price $\bar{U}_t$ in place of $\hat{U}_t$.

**Theorem 5.5.** The robust XVA admits the explicit representation given by

$$\text{rXVA}_t = \bar{U}_t^* \mathbb{I}_{\{t < \tau \wedge \gamma_1\}} + \left(\tilde{\theta}_C(\tilde{V}_{\tau_C}, M_{\tau_C}) \mathbb{I}_{\{\tau_C < \tau \wedge \gamma_1 \wedge T\}} + \tilde{\theta}_I(\tilde{V}_{\tau_I}, M_{\tau_I}) \mathbb{I}_{\{\tau_I < \tau \wedge \gamma_C \wedge T\}}\right) \mathbb{I}_{\{\tau \wedge \gamma_1 \leq T\}},$$  \hspace{1cm} (35)

and the corresponding super-replicating strategies for rXVA are given by

$$\xi_t^1, \xi_t^C = \frac{\bar{U}_t^*}{\mathbb{P}_t^-} \mathbb{I}_{\{t < \tau \wedge \gamma_1\}},$$

$$\xi_t^I = \frac{L_I(\tilde{V}_t - M_{\tau_I})^+ + \bar{U}_t^*}{\mathbb{P}_t^-} \mathbb{I}_{\{t \leq \tau \wedge \gamma_1\}},$$

$$\xi_t^C = -\frac{L_C(\tilde{V}_t - M_{\tau_C})^- + \bar{U}_t^*}{\mathbb{P}_t^-} \mathbb{I}_{\{t \leq \tau \wedge \gamma_1\}},$$

$$\psi_t^{m, \ast} = -\frac{M_{t_I^-}}{B_t^{I_m}} \mathbb{I}_{\{t < \tau \wedge \gamma_1\}},$$

$$\psi_t^{I, \ast} = -\frac{2\bar{U}_t^* + L_C(\tilde{V}_t - M_{\tau_C})^- - L_I(\tilde{V}_t - M_{\tau_I})^+ - M_{t_I^-}}{B_t^{I_I}} \mathbb{I}_{\{t < \tau \wedge \gamma_1\}}.$$  \hspace{1cm} (36)

**Proof.** The proof consists of two parts. In the first part, we verify that the expression of rXVA given in Eq. (35) is the smallest super-replicating price. In the second part, we show that the strategy given in (36) is a super-replicating strategy. This requires showing that the implementation of this strategy does not require any cash infusion, and that the wealth process controlled by this strategy is exactly the rXVA process.

Define $XVA_t^\mu := V_t^\mu - \tilde{V}_t$ for $\mu \in \mathbb{F}$, $\mu_C \leq \mu \leq \bar{p}_C$ and $XVA_t^{(\mu_C)} := V_t^{(\mu_C)} - \tilde{V}_t$, where we recall that $V_t^\mu$ is the valuation process of the replicating portfolio obtained by setting the counterparty bond rate equal to $\mu$; see also the discussion before Eq. (10). First, note that $XVA_t^\mu \geq XVA_t^{(\mu_C)}$. This follows directly from Theorem 5.4, which provides a comparison result for the term $\hat{U}_t$ appearing on the right hand side of the XVA expression (25). Therein, it is enough to observe that the risk-neutral default intensity is just $\mu - r_D$, and notice that the two closeout terms are just independent of the rate $\mu$. Hence, the right hand side of Eq. (35) is smaller than the left hand side: the latter represents a specific $\mathbb{F}$-predictable intensity process satisfying the boundary conditions, while the former is the supremum over all such intensity processes. This shows that the left side of Eq. (35) is less or equal than the right side. To show the reverse inequality, i.e., that the left side of Eq. (35) is greater or equal than the right side, we note that the family $(XVA_t^\mu)_{\mu \in \mathbb{F}, \mu \in [\mu_C, \bar{p}_C]}$ is directed upwards, i.e., for $\mu', \mu'' \in \mathbb{F}$, $\mu_C \leq \mu' \leq \mu'' \leq \bar{p}_C$,
there exists a process $\mu''' \in \mathbb{F}, \mu_C \leq \mu''' \leq \mu_C$, such that $XVA^\mu \vee XVA^\mu'' \leq XVA^\mu'''$. Indeed, setting $A := \{ \omega \in \Omega : XVA^\mu_t \geq XVA^\mu''_t \}$ we can define $\mu'''$ directly by setting $\mu'''(s) := \mu'(s)1_A + \mu''(s)1_A^c$ for $s \geq t$ and $\mu'''(s) = 0$, for $0 \leq s < t$. Such a process is clearly $\mathcal{F}_s$-measurable because $A$ is $\mathcal{F}_t$-measurable. As the essential supremum of an upward directed set can be written as a monotone limit (see Föllmer et al., 2004, Theorem A.32), \( \lim_{n \to \infty} XVA^{\mu(n)} = rXVA \). Thus, as the countable union of nullsets is still a nullset we have that, for all \( t \), \( rXVA_t \) is smaller or equal than the right side of Eq. (35).

Next, we provide the expressions for the super-replicating strategies. These are derived by replacing\( \tilde{U}_t \) with \( \tilde{U}_t^* \) into equations (27)-(31). Using the replicating strategies defined in (36), we obtain that, on the set \( \{ t < \tau \} \), the value of the replicating portfolio at time $t$ is

$$
\xi_{u}^{1,*} P_{u}^1 + \xi_{t}^{1,*} P_{t}^1 + \xi_{t}^{C,*} P_{t}^C + \xi_{t}^{f,*} B_{t}^r - \psi_{t}^{m,*} B_{t}^{m} = \tilde{U}_t^*.
$$

On the set \( \{ t < \tau \} \), the change in value of the portfolio is

$$
\xi_{u}^{1,*} dP_{u}^1 + \xi_{t}^{1,*} dP_{t}^1 + \xi_{t}^{C,*} dP_{t}^C + \xi_{t}^{f,*} dB_{t}^r - \psi_{t}^{m,*} dB_{t}^{m}
= \left( \mu\tilde{U}_t^* + \left( L(\tilde{V}_t - M_{t-}) + \tilde{U}_t^* \right) \right) \mu + \left( \tilde{U}_t^* - L_C(\tilde{V}_t - M_{t-}) \right) \mu_C
+ r_m^t M_t^+ + r_m^t M_t^+ + r_f^t \left( -2\tilde{U}_t^* + L_C(\tilde{V}_t - M_{t-}) - L_I(\tilde{V}_t - M_{t-})^+ - M_{t-} \right)
+ r_f^t \left( -2\tilde{U}_t^* + L_C(\tilde{V}_t - M_{t-}) - L_I(\tilde{V}_t - M_{t-})^+ - M_{t-} \right) dt.
$$

Additionally, for the replicating strategy (27)-(31) to be self-financing, we need to include the cash flow

$$
\left( r_f(\xi_{t}^{f,*} - r_D) \right) L_1 dt.
$$

The presence of this cash flow is due to the fact that the clean valuation $\hat{V}$ is computed using the publicly available discount rate $r_D$, while the private valuation $V$ is obtained using the funding rate $r_f$. Such a cash flow needs to be accounted for in the implementation of the super-replicating strategy. Taken together, equations (38) and (39) describe the change in value of the super-replicating portfolio. Next, we compare it with the change in value of the robust XVA process given by

$$
\tilde{U}_t^* = \left( (r_D + h_f^t)(L(I(\tilde{V}_t - M_{t-}) + \tilde{U}_t^*) - (\mu_C)(\tilde{V}_t, M_t, \tilde{U}_t^*) (L_C(\tilde{V}_t - M_{t-}) - \tilde{U}_t^*)
+ \mu_1 \tilde{U}_t^* + r_m^{t}M_t^+ + r_m^{t}M_t^- + r_f^t \left( -2\tilde{U}_t^* + L_C(\tilde{V}_t - M_{t-}) - L_I(\tilde{V}_t - M_{t-})^+ + L_1 - M_{t-} \right)
+ r_f^t \left( -2\tilde{U}_t^* + L_C(\tilde{V}_t - M_{t-}) - L_I(\tilde{V}_t - M_{t-})^+ + L_1 - M_{t-} \right) - r_D L_1 \right) dt.
$$

Using the fact that

$$
(\mu_C)(\tilde{V}_t, M_t, \tilde{U}_t^*) - \tilde{U}_t^* \geq 0,
$$

it follows that (38) together with (39) dominate (40) from above, i.e.,

$$
\xi_{u}^{1,*} dP_{u}^1 + \xi_{t}^{1,*} dP_{t}^1 + \xi_{t}^{C,*} dP_{t}^C + \xi_{t}^{f,*} dB_{t}^r - \psi_{t}^{m,*} dB_{t}^{m} + \left( r_f(\xi_{t}^{f,*} - r_D) \right) L_1 dt \geq d\tilde{U}_t^*.
$$

The above computations were done on the set \( \{ t < \tau \} \). At the stopping time $\tau$ it can be easily checked that both $\tilde{U}$ and the super-replicating portfolio are zero. Together with (41) and Theorem 5.4, it follows that the super-replicating portfolio dominates $\tilde{U}$ for all times $t$. 

\[ \Box \]
We notice that if we use the robust super-replicating strategies given in (36) and start with an initial capital rXVA0, then there will be no tracking error in the sense of El Karoui et al. (1998). In other words, the error committed for implementing the robust strategy (ξi, ξi, ξi, ξi, ψi, ψi) in the real market (where the return rate of the counterparty bond is µC) instead of the robust market model (where the return rate of the counterparty bond is (µC)*) is zero. This may be understood as follows: Eq. (37) shows that the value of the super-replicating portfolio is always rXVA. However, until the earliest among the default time of the counterparty, investor, or maturity of the CDS contract, whichever comes first, the super-replicating portfolio keeps generating profits because the change in the value of the super-replicating portfolio is greater than the change in the value of the CDS contract, as shown in (41). In other words, during a time interval dt, the investor pockets an extra cash \( \left( (\mu_C)^* (V_t, M_t, \bar{U}_t) - \mu_C \right) (\tilde{\theta}_C(V_t, M_t) - \bar{U}_t^*) dt \) at any time prior to the end of the replication strategy.

The robust strategies depend only on the XVA price process and the bond prices, and are independent of the default intensity \( h_C^\theta \), the value of which is unknown to the investor.

![Figure 2: We use the following benchmark parameters: \( r_f^\pm = r_D = 0.001 \), \( \alpha = \beta = 0 \), \( T = 3 \), \( L_I = L_C = 0.5 \), \( S_1 = 2 \), \( h_1(t) = 0.11_{t<1} + 0.33_{1<t<T} \), \( L_I = 10 \), \( \mu_I = 0.2001 \), \( \mu_C = 0.2501 \), \( \mu_C = 0.1501 \), \( \mu_C = 0.2001 \). Left panel: Plot of \( \tilde{u} \) (solid), \( \hat{u}^* \) (dashed) and \( \tilde{u}^* \) (dotted) as a function of time. Right panel: Plot of \( \tilde{\theta}_C(\hat{v}, 0) \) (dash-dotted), \( -\tilde{\theta}_I(\hat{v}, 0) \) (dotted) and \( \tilde{u} \) (solid) as a function of time. In the left panel, the default intensity at which we switch between the sub-and super-solutions is the crossing point of the dashed and dotted lines with the x-axis, that occurs at approximately \( t = 2.67 \). In the right panel, the third party valuation \( \hat{v} \) becomes positive at approximately \( t = 2.67 \).

In the case of zero margins, it follows directly from Eq. (20) that the third party valuation \( \hat{v} = \hat{v}^+ - \hat{v}^- \) may be expressed in terms of the closeout value, and given by \( -\tilde{\theta}_C(\hat{v}, 0) - \tilde{\theta}_I(\hat{v}, 0) \). Hence, we deduce from the right panel of Figure 2 that the third party valuation is negative prior to \( t = 1.83 \), and positive for \( t > 1.83 \). Figure 2 also shows that the super-replicating strategy is non-trivial in the sense that it is not monotone in the default intensity. As it can be seen from the right panel of Figure 2, the quantity \( \tilde{\theta}_C(\hat{v}, 0) - \hat{u} \) is zero at \( t_0 \approx 2.67 \), non-negative for \( t < t_0 \), and strictly negative for \( t > t_0 \). This implies that \( (\mu_C)^* = \mu_C \) prior to time \( t_0 \) and while \( \tilde{\theta}_C(\hat{v}, 0) - \hat{u} \geq 0 \), whereas after time \( t_0 \), \( (\mu_C)^* \leq \mu_C \), because we then have \( \tilde{\theta}_C(\hat{v}, 0) - \hat{u} \leq 0 \). In other words, prior to \( t_0 \) the trader will use the largest value of the bond rate \( \mu_C \) for her super-replicating portfolio because the jump of the super-replicating portfolio to the close-out value when the counterparty defaults, given by \( \tilde{\theta}_C(\hat{v}, 0) - \hat{u} \), is positive. After time \( t_0 \), the trader will choose the smallest value \( \mu_C \) of the bond rate because this
jump would be negative. This is directly visible from the right panel of Figure 2, because the dash-dotted line dominates the solid one. This analysis highlights a fundamental difference with respect to standard credit risk settings, that often ignore collateralization and close-out terms, or models for XVA in which collateralization and close-out value depend on the trader’s valuation process \( V \) itself as in Nie and Rutkowski (2016). In these cases, the price of the derivative is monotone in the default intensity, while in our setting the value of the super-replicating portfolio does not necessarily have this monotonicity property. This is due to the fact that the collateralization and closeout process are exogenous, i.e., they depend on the external valuation \( \hat{V} \) of the third party, rather than on the value \( V \) of the super-replicating portfolio.

\section{5.3 Computation of Margins}

We develop an explicit expression for the initial margins when the two parties trade \( \gamma \) units of a single name credit default swap contract. Initial margins are determined using the value-at-risk criterion, and need to be computed under the physical measure \( \mathbb{P} \) as opposed to the valuation measure \( \mathbb{Q} \). By the definition of \( VaR \), we have

\[
IM_t(\gamma) = \beta VaR_q \left( \gamma \hat{V}_{(t+\delta)\wedge T} - \gamma \hat{V}_t \mid \tau \wedge \tau(N) > t \right)^+
\]

\[
= \beta \inf \left\{ K \in \mathbb{R}_{>0} : \mathbb{P}[\gamma \hat{V}_{(t+\delta)\wedge T} + \gamma \hat{V}_t > -K \mid \tau > t] \geq 1 - q \right\}. \tag{42}
\]

Thus, differently from the variation margin \( VM(\gamma) = \alpha \gamma \hat{V}_t \mathbb{1}_{\{\tau \wedge \tau(N) > t\}} \) that is linear in \( \gamma \), the initial margin \( IM(\gamma) \) is only positively homogeneous in \( \gamma \). We will therefore distinguish the cases \( \gamma = 1 \) and \( \gamma = -1 \). Note first that

\[
\gamma \left( \hat{V}_{(t+\delta)\wedge T} - \hat{V}_t \right) = -\gamma \begin{cases} S_1 ((t+\delta) \wedge T - t) & \text{if } \tau \geq (t + \delta) \wedge T, \\ -L_1 + S_1 (\tau - t) & \text{otherwise}. \end{cases}
\]

The case \( \gamma = 1 \) is less frequently observed in practice. We typically expect \( L_1 > S_1 T \), as a protection buyer is unlikely to pay more than what he would receive in the event of a default (notice that \( S_1 T \) is the maximum payment the buyer would make). In this case, the exposure of the protection seller to the protection buyer would be negative, resulting in negative \( VaR \). In the case \( \gamma = -1 \), we obtain

\[
\mathbb{P}[ - \hat{V}_{(t+\delta)\wedge T} + \hat{V}_t > -K \mid \tau > t] = \frac{\mathbb{P}[ - \hat{V}_{(t+\delta)\wedge T} + \hat{V}_t > -K]}{\mathbb{P}[\tau > t]}
\]

\[
= \frac{\int_{t}^{(t+\delta)\wedge T} \left( K - L_1 + S_1 (u - t) \right) e^{-h^1_t(u)} du}{\int_t^{\infty} e^{-h^1_t(u)} du}, \tag{43}
\]

Because the right hand side of Eq. (43) is continuous and increasing in \( K \), the inequality (42) that characterizes the initial margins becomes an equality, and thus the \( VaR \) can be numerically evaluated.

\begin{example}
Assume constant default intensities and \( t < T - \delta \). To calculate the initial margin \( IM_{T}^\gamma \) for \( \gamma = -1 \), we note that

\[
\mathbb{P}[ - \hat{V}_{(t+\delta)\wedge T} + \hat{V}_t > -K \mid \tau > t] = \frac{\mathbb{P}[ - \hat{V}_{(t+\delta)\wedge T} + \hat{V}_t > -K]}{\mathbb{P}[\tau > t]}
\]

\[
= \frac{\int_{t}^{(t+\delta)\wedge T} \left( K - L_1 + S_1 (u - t) \right) e^{-h^1_t(u)} du}{\int_t^{\infty} e^{-h^1_t(u)} du} = 1 - e^{-h^1_t \left( \frac{L_1 - K}{S_1} \right) \wedge \delta}. \tag{44}
\]
\end{example}
Therefore, the value of $K$ solving the above equation, i.e., the initial margin, is explicitly given by

$$IM_t(-1) = \begin{cases} \beta \left( L_1 + S_1 \log q \right) & \text{if } q > e^{-h \delta}, \\ 0 & \text{otherwise}. \end{cases}$$ (45)

The initial margin formula (45) has a direct economic interpretation. First, we notice that the term multiplying the spread $S_1$ is negative, because the value-at-risk level $q$ is between 0 and 1 and hence $\log q$ is negative. Thus, when the initial margin is nonzero, it is affine both in the loss rate and the CDS spread, increasing in the loss rate and decreasing in the CDS spread. This is intuitive: the protection seller increases the margin requirements if he has to make a larger payment at the credit event, and decreases the requirement if the running spread premim received from the protection buyer is higher. Moreover, the required margin is increasing in the default intensity of the reference entity and, in the limiting case of an infinite default intensity, it converges to the product $L_1 \beta$ of loss and collateralization rates. This is in line with economic intuition: as the credit event becomes more likely to occur, the protection seller asks the buyer to pay exactly the amount he would receive at the credit event. Finally, the value of initial margins is decreasing in the value-at-risk level and linearly increasing in the collateralization rate.

### 5.4 Credit Swap Portfolios

In this section, we generalize the analysis conducted in the previous sections to a portfolio of single name credit default swaps, each referencing a different entity. To capture direct default contagion, we let the default intensities of surviving entities depend on past defaults. Throughout the section, we use the superscript $(J)$, $J \subset \{1, ..., N\}$, to denote the set of already defaulted entities, while the other entities in $J^c := \{1, ..., N\}\setminus J$ are all alive. For instance, $V^{(J)}$ denotes the replicating process of the CDS portfolio when all the entities in the set $J$ have defaulted, and all entities in the set $J^c$ have not.

We denote by $\tau^{(J)}$ the last default time of a reference entity in $J$ (i.e., $\tau^{(J)} = \max_{j \in J} \tau_j$), assuming $\max_{j \in J} \tau_j < \tau_i$, $i \notin J$, and for $i \notin J$ we use $\tau_i^{(J)}$ to denote the default time of the $i$-th reference entity in the economic scenario where all reference entities in $J$ have defaulted.

First, we study the dynamics of the third party valuation process $\hat{V}$. Note that if all entities have defaulted, then $\hat{V}^{(1, ..., N)} = 0$. The case when all entities except for $i$ have already defaulted, that is $J = \{1, ..., N\}\setminus \{i\}$ (in this case $\tau_i^{(J)} = \tau^{(1, ..., N)}$), is analogous to the case of a single CDS contract, whose price process has been given in Eq. (17). Hence

$$-d\hat{V}_t^{(J)} = \left( -r_D \hat{V}_t^{(J)} - S_i \right) dt - \hat{Z}_t^{i^{(J)}} \omega_t^{i^{(J)}, Q},$$

$$\hat{V}_t^{(J)}_{\tau_i^{(J)} \wedge T} = L_i \mathbb{I}_{\{\tau_i^{(J)} < T\}}.$$ (46)

Next, we provide an inductive relation which relates the investor’s wealth price process in the state where all entities in $J$ have defaulted, to that in the state where the reference entity $i \notin J$ additionally defaults. The base case $|J| = N - 1$ has been given in (46). For the case $|J| < N - 1$, we obtain

$$-d\hat{V}_t^{(J)} = -\left( r_D \hat{V}_t^{(J)} - \sum_{k \in J^c} S_k \right) dt - \sum_{k \in J^c} \hat{Z}_t^{k^{(J)}} \omega_t^{k^{(J)}, Q},$$

$$\hat{V}_t^{(J)}_{T \wedge \min_{j \in J^c} \tau_j^{(J)}} = \sum_{k \in J^c} \left( L_k + \hat{V}_t^{(k)^{(J))} \mathbb{I}_{\{\tau_k^{(J)} = \min_{j \in J^c} \tau_j^{(J)}\}} \right) \mathbb{I}_{\{\tau_k^{(J)} < T\}}.$$ (47)

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The price process and the replicating strategy are then obtained by considering all possible subsets of defaulted entities, leading to
\[ \hat{V}_t = \sum_{J \in 2^{(1, \ldots, N)}} \hat{V}_t^{(J)} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}, \]
\[ \hat{Z}_t^i = \sum_{J \in 2^{(1, \ldots, N)} \setminus \{i\}} \hat{Z}_t^{i{(J)}} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}. \]

The collateral process \( M \) is still given by Eq. (6) along with the above expression for \( \hat{V} \).

Next, we use a similar inductive argument to define the \( V := (V_t)_{t \geq 0} \) process. Clearly, \( V^{(1, \ldots, N)} = 0 \). Consider now the state when all but entity \( i \) have defaulted, that is \( J = \{1, \ldots, N\} \setminus \{i\} \) (and \( \tau^{(J)} = \tau^{((1, \ldots, N))} \)). The corresponding expression to (15) in the multi-name case is then given by

\[ -dV_t^{(J)} = f(t, V_t^{(J)}, Z_t^{k,(J)}, Z_t^{I,(J)}, Z_t^{C,(J)}, M^{(J), J}) dt - Z_t^{I,(J)} d\pi_t, M^{(J)} - Z_t^{C,(J)} d\pi_t, C; Q, \]

\[ V_t^{(J), \tau_{\wedge \tau^{(J)}}} = L_t \mathbb{1}_{\{\tau^{(J)} < \tau \wedge T\}} + \theta_t \left( \hat{V}_t^{(J)}, M^{(J)}_{\tau^{(J)}} \mathbb{1}_{\{\tau < \tau^{(J)} \wedge \tau_C \wedge T\}} + \theta_C \left( \hat{V}_t^{(J)}, M^{(J)}_{\tau^{(J)}} \right) \mathbb{1}_{\{\tau < \tau^{(J)} \wedge \tau_I \wedge T\}} \right), \]

where the representation of \( f \) has a similar structure to the single name case treated in (16), and is given by

\[ f(t, v, z, z^I, z^C, M, J) := \left( r^+_f (v + z + z^I + z^C - M) + r^-_f (v + z + z^I + z^C - M) - r_D z - r_D z^I - r_D z^C + r^+_m M^+ + r^-_m M^- + \sum_{k \in J^c} S_k. \right). \]

Similar to Eq. (47), the wealth replicating process in the state where all reference entities in \( J \) have defaulted is related to the state where the additional entity \( i \notin J \) defaults:

\[ -dV_t^{(J)} = f(t, V_t^{(J)}, \sum_{k \in J^c} Z_t^{k,(J)}, Z_t^{I,(J)}, Z_t^{C,(J)}, M^{(J), J}) dt \]

\[ - \sum_{k \in J^c} Z_t^{k,(J)} d\pi_t, k; Q - Z_t^{I,(J)} d\pi_t, I; Q - Z_t^{C,(J)} d\pi_t, C; Q, \]

\[ V_t^{(J), \tau_{\wedge \min_{j \in J^c} \tau^{(J)}}} = \sum_{k \in J^c} \left( L_t \mathbb{1}_{\{\tau^{(k)} \wedge \tau^{(J)} \leq T\}} \mathbb{1}_{\{\tau^{(k)} \wedge \tau^{(J)} \leq T\}} \right) \mathbb{1}_{\{\tau^{(k)} \wedge \tau^{(J)} \leq T\}} \]

\[ + \theta_t \left( \hat{V}_t^{(J)}, M_{\tau^+} \mathbb{1}_{\{\tau^{(J)} \wedge \tau^+ \wedge \tau_C \wedge T\}} + \theta_C \left( \hat{V}_t^{(J)}, M_{\tau^+} \right) \mathbb{1}_{\{\tau^{(J)} \wedge \tau^+ \wedge \tau_I \wedge T\}} \right). \]

Altogether, we obtain

\[ V_t = \sum_{J \in 2^{(1, \ldots, N)}} V_t^{(J)} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}, \]
\[ Z_t^i = \sum_{J \in 2^{(1, \ldots, N)} \setminus \{i\}} Z_t^{i{(J)}} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}, \]
\[ Z_t^I = \sum_{J \in 2^{(1, \ldots, N)}} Z_t^{I{(J)}} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}, \]
\[ Z_t^C = \sum_{J \in 2^{(1, \ldots, N)}} Z_t^{C{(J)}} \mathbb{1}_{\{\tau^{(J) \wedge \tau_C \wedge \tau_I < T \leq \min_{k \in J \setminus \{i\}} \tau^{(J) \wedge \tau_C \wedge \tau_I < T}\}}, \]
Proceeding along the lines of Section 5, we can obtain a BSDE for the XVA process specified in Eq. (9):

$$-d\text{XVA}^{(j)}_t = \tilde{f}(t, \text{XVA}^{(j)}_t, \sum_{k \in J^c} \tilde{Z}^{k,(j)}_t, \tilde{Z}^{I,(j)}_t, \tilde{Z}^{C,(j)}_t; M^{(j)}_t, J) \, dt$$

$$- \sum_{k \in J^c} \tilde{Z}^{k,(j)}_t d\omega_t^{k,(j),Q} - \tilde{Z}^{I,(j)}_t d\omega_t^{I,Q} - \tilde{Z}^{C,(j)}_t d\omega_t^{C,Q},$$

$$\text{XVA}^{(j)}_{\tau \wedge \min_{j \in J^c} \tau_j^{(j)}} = \sum_{k \in J^c} \left( L_k + \text{XVA}^{(k) \cup J}_t \right) \mathbb{1}_{\{ \tau_k^{(j)} = \min_{j \in J^c} \tau_j^{(j)} \wedge \tau_T \}} \mathbb{1}_{\{ \tau_k^{(j)} < T \}}$$

$$+ \tilde{\theta}_t \left( \tilde{V}^{(j)}(t), M_{\tau_-} \right) \mathbb{1}_{\{ \tau_j^{(j)} < \min_{j \in J^c} \tau_j^{(j)} \wedge \tau_T \}}$$

$$+ \tilde{\theta}_C \left( \tilde{V}^{(j)}(t), M_{\tau_-} \right) \mathbb{1}_{\{ \tau_C < \min_{j \in J^c} \tau_j^{(j)} \wedge \tau_T \}},$$

where \( \tilde{\theta}_C \) and \( \tilde{\theta}_t \) are given in (20), \( \tilde{Z}^{k,(j)}_t, \tilde{Z}^{I,(j)}_t \) and \( \tilde{Z}^{C,(j)}_t \) are defined as

$$\tilde{Z}^{k,(j)}_t := Z^{k,(j)}_t - \tilde{Z}^{l,(j)}, \quad i \in J^c, \quad \tilde{Z}^{I,(j)}_t := Z^{I,(j)}_t, \quad \tilde{Z}^{C,(j)}_t := Z^{C,(j)}_t,$$

and

$$\tilde{f}(t, xva, \tilde{z}, \tilde{z}^l, \tilde{z}^C; M, J) := - \left( r^+ \left( xva + \tilde{z} + \tilde{z}^l + \tilde{z}^C + \sum_{k \in J^c} L_k - M_t \right) \right)^{+}$$

$$- \left( r^- \left( xva + \tilde{z} + \tilde{z}^l + \tilde{z}^C + \sum_{k \in J^c} L_k - M_t \right) \right)^{-}$$

$$- r_D \tilde{z} - r_D \tilde{z}^l - r_D \tilde{z}^C + r^+_m M^+_t - r^-_m M^-_t - r_D \sum_{k \in J^c} L_k,$$

where the terminal condition is \( \text{XVA}_T^{(1, \ldots, N)} = 0 \). The BSDE in the reduced filtration \( \mathbb{F} \) can be obtained analogously to (22), and is given by

$$-d\tilde{U}^{(j)}_t = \tilde{g}(t, \tilde{U}^{(j)}_t, \sum_{k \in J^c} \tilde{U}^{(k) \cup J}_t, \sum_{k \in J^c} \tilde{h}^Q_k \tilde{U}^{(k) \cup J}_t; \tilde{V}^{(j)}, M^{(j)}_t, J) \, dt,$$

$$\tilde{U}^{(j)}_T = 0,$$

with

$$\tilde{g}(t, \tilde{u}, \tilde{o}, u; \tilde{V}, M, J) = h^Q_t \left( \tilde{\theta}_t \left( \tilde{V}^{(j)}(t), M^{(j)}_t \right) - \tilde{u} \right) + h^C_t \left( \tilde{\theta}_C \left( \tilde{V}^{(j)}(t), M^{(j)}_t \right) - \tilde{u} \right) + \left( \tilde{u} - \sum_{k \in J^c} h^Q_k \tilde{u} \right)$$

$$+ \tilde{f}(t, \tilde{u}, \tilde{o} - |J^c| \tilde{u}, \tilde{\theta}_t \left( \tilde{V}^{(j)}(t), M^{(j)}_t \right) - \tilde{u}, \tilde{\theta}_C \left( \tilde{V}^{(j)}(t), M^{(j)}_t \right) - \tilde{u}; M).$$

The starting point for the recursion is set to \( \tilde{U}^{(1, \ldots, N)}_t = 0 \).

**Remark 5.7.** For large portfolios, i.e., those referencing a high number \( N \) of entities, this system of ODEs is computationally intractable. A solution to the ODE (55) would need to be obtained for each subset of \( \{1, \ldots, N\} \), that is a total of \( 2^N \) solutions need to be computed. This system becomes tractable only if the reference entities have identical characteristics (spreads, loss rates and default
intensities), and the default intensities depend only on the number of occurred defaults, but not on the identity of the defaulted entities, i.e., \( h_i^Q = h_i^Q(t, |J|) \), \( i = 1, ..., N \). In this case, the complexity grows linearly and it is required to compute \( N \) ODE solutions. This assumption has been used to calibrate models of direct default contagion for pricing. For instance, Frey and Backhaus (2004) split firms into groups, each defining a specific default risk profile. Because firms belonging to the same group are exchangeable, they naturally consider the above parameterization for the default intensity of each firm in the portfolio. The dependence structure between ODEs has similar characteristics to that arising in a binomial tree. Computations on a non-recombining tree are generally prohibitively expensive, and thus recombining trees are usually used.

Assume that \( h_i^Q \), \( i = 1, ..., N \), are all piecewise (deterministic) continuous. The extension of the theorems developed in Section 5.2 for the case of a single CDS to the case of portfolios referencing multiple entities is straightforward. For notational consistency, denote \( \hat{\varphi}(t) = \hat{V}^i_{T-t}, \hat{u}(t) = \hat{V}^i_{T-t} \) for all \( J \subset \{1, ..., N\} \). The following proposition is the multi-dimensional extension of Proposition 5.3. Its proof uses exactly the same arguments and is omitted here.

**Proposition 5.8.** There exists a unique (piecewise smooth) solution to the system of ODEs:

\[
\partial_t \hat{\varphi}(t) = -r_D \hat{\varphi}(t) + \sum_{i \in J^c} S_k + \sum_{k \in J} \left( L_k + \hat{\varphi}(k \cup J) - \varphi(t) \right) h_i^Q, \quad J \subset \{1, ..., N\},
\]

\[\hat{\varphi}(0) = 0,\]

\[
\partial_t \hat{u}(t) = \hat{g} \left( t, \hat{u}(t), \sum_{k \in J^c} \hat{u}(k \cup J), \sum_{k \in J} h_i^Q \hat{u}(k \cup J); \hat{u}(t), m, J \right), \quad J \subset \{1, ..., N\},
\]

\[\hat{u}(0) = 0.\]

Next, we present the multi-dimensional extension of Theorem 5.4. The proof presents an additional induction step compared with the proof of Theorem 5.4, and the details are reported in the appendix.

**Theorem 5.9 (Comparison Theorem).** Let \( J \subset \{1, ..., N\} \). Assume in addition to Assumption 4.4 that \( r_f < \min_{i \in \{1, ..., N\}} \mu_i \wedge \mu_C \). Moreover, assume that there exists \( \overline{\mu}_C \geq \mu_C > r_D \) such that \( \overline{\mu}_C \geq \mu_C^Q(t) \geq \mu_C \), and let \( \hat{u}(t) \) be the solution of ODE (57). Let

\[
(\mu_C)^*(\hat{v}, m, \hat{u}) = \overline{\mu}_C \mathbb{1}_{\hat{u} = 0} + \mu_C \mathbb{1}_{\hat{u} < 0},
\]

and define \( g^* \) and \( g_* \) plugging \( (h_C^Q)^* \) and \( (h_C^Q)_* \) into \( \hat{g} \) given in Eq. (55), i.e.,

\[
g^*(t, \hat{v}, \hat{u}, \hat{w}; \hat{v}, m, J) := h_i^Q \left( \hat{\theta}_I(\hat{v}(t), m(t)) - \hat{u}(t) \right) + ((\mu_C)^*(\hat{v}(t), m(t), \hat{u}(t)) - r_D) \left( \hat{\theta}_C(\hat{v}(t), m(t)) - \hat{u}(t) \right) + \left( \hat{v} - \sum_{i \in J^c} h_i^Q \hat{u}(t) \right) + \hat{f}(t, \hat{u}(t), \hat{v} - |J^c| \hat{u}(t), \hat{\theta}_I(\hat{v}(t), m(t)) - \hat{u}(t), \hat{\theta}_C(\hat{v}(t), m(t)) - \hat{u}(t); \hat{v}, m, J),
\]

\[
g_*(t, \hat{v}, \hat{u}, \hat{w}; \hat{v}, m, J) := h_i^Q \left( \hat{\theta}_I(\hat{v}(t), m(t)) - \hat{u}(t) \right) + ((\mu_C)_*(\hat{v}(t), m(t), \hat{u}(t)) - r_D) \left( \hat{\theta}_C(\hat{v}(t), m(t)) - \hat{u}(t) \right) + \left( \hat{v} - \sum_{i \in J^c} h_i^Q \hat{u}(t) \right) + \hat{f}(t, \hat{u}(t), \hat{v} - |J^c| \hat{u}(t), \hat{\theta}_I(\hat{v}(t), m(t)) - \hat{u}(t), \hat{\theta}_C(\hat{v}(t), m(t)) - \hat{u}(t); \hat{v}, m, J).
\]
Finally, let \( \tilde{u}^{(J),*} \) be the solution to ODE (57), but with \( \tilde{g} \) replaced by \( g^* \), that is
\[
\partial_t \tilde{u}^{(J),*} = g^* \left( t, \tilde{u}^{(J),*}, \sum_{k \in J} \tilde{u}^{(k) \cup J,*,*}_{k}, \sum_{k \in J} \tilde{u}^{(k) \cup J,*,*}_{k}; \tilde{u}^{(J),*}, m, J \right),
\]
(58)
and similarly, let \( \tilde{u}^{(J),*}_a \) be the solution of ODE (57) where we replace \( \tilde{g} \) with \( g_* \). Then \( \tilde{u}^{(J),*}_a \leq \tilde{u}^{(J),*} \leq \tilde{u}^{(J),*} \).

It now remains to find the super-replicating strategy for the robust XVA process. Following similar arguments to those used above, the strategy will be obtained by pasting together the various quantities associated with different subsets \( J \) of defaulted entities.

**Theorem 5.10.** The robust XVA can be represented explicitly by
\[
rXVA_t = \sum_{J \in 2^{\{1, \ldots, N\}}} \bar{U}^{(J),*}_t \mathbb{1}_{\{r^{(J),*} \cap \tau \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap T\}}
\]
\[
+ \left( \bar{\theta}_C(\bar{V}_t, M_{\tau T}) \mathbb{1}_{\{\tau \cap C < \min_{j \in J} \tau^{(J),*}_j \cap \tau \cap T\}} + \bar{\theta}_I(\bar{V}_t, M_{\tau T}) \mathbb{1}_{\{\tau \cap I < \min_{j \in J} \tau^{(J),*}_j \cap \tau \cap T\}} \right) \mathbb{1}_{\{\tau \cap T \leq t \leq T\}},
\]
(59)
where the process \( \bar{U}^{(J),*}_t := \tilde{u}^{(J),*}_a(T - t) \). Define
\[
\xi_t^{i,(J),*} = \frac{\tilde{u}^{(J),*}_t - \bar{U}^{(1),*}_t}{P_t^{-}} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
(60)
\[
\xi_t^{l,(J),*} = \frac{L \bar{I}(\bar{V}_t - M_{\tau T})^+ + \bar{U}^{(J),*}_t}{P_t^{-}} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
\[
\xi_t^{c,(J),*} = \frac{-C(\bar{V}_t - M_{\tau T})^- - \bar{U}^{(J),*}_t}{P_t^{-}} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
\[
\xi_t^{f,(J),*} = \frac{-\tilde{u}^{(J),*}_t - \sum_{i \in J} (\tilde{u}^{(J),*}_t - \bar{U}^{(1),*}_t)}{B_t^I} + L \bar{I}(\bar{V}_t - M_{\tau T})^- - L \bar{I}(\bar{V}_t - M_{\tau T})^+ - M_{\tau T} \times \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}}.
\]
The super-replicating strategies for rXVA are obtained from the above conditional strategies as
\[
\xi_t^{i,(J),*} = \sum_{J \in 2^{\{1, \ldots, N\}\ Set(J)}} \xi_t^{i,(J),*} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
(61)
\[
\xi_t^{l,(J),*} = \sum_{J \in 2^{\{1, \ldots, N\}\ Set(J)}} \xi_t^{l,(J),*} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
\[
\xi_t^{c,(J),*} = \sum_{J \in 2^{\{1, \ldots, N\}\ Set(J)}} \xi_t^{c,(J),*} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]
\[
\xi_t^{f,(J),*} = \sum_{J \in 2^{\{1, \ldots, N\}\ Set(J)}} \xi_t^{f,(J),*} \mathbb{1}_{\{\tau \cap C \cap \tau \cap I \cap T < t \leq \min_{k \in J} \tau^{(J),*}_k \cap \tau \cap I \cap T\}},
\]

\[
\psi_t^{m,*} = -\frac{M_{\tau T}}{B_t^{-}} \mathbb{1}_{\{t \leq \tau \cap I \cap T\}},
\]
(62)
Proof. The proof that rXVA dominates XVA\(^\mu\) for any \(\mu_C \leq \mu \leq \mu_C\) is done in the same way as in the proof of Theorem 5.5. To prove that the super-replicating strategy is given by equations (61) and (62), fix \(J \subset \{1, \ldots, N\}\). Then the value of the portfolio associated with this strategy at time \(t\) on the set \(\{\tau^J \land \tau_C \land \tau_T \land T < t < \min_{k \in J^C} \tau_k^J \land \tau_C \land \tau_T \land T\}\) is

\[
\sum_{i \in J^c} \xi_u^{i,*} P_u^i + \xi_t^{f,(J),*} P_t^i + \xi_t^{C,(J),*} P_C^i + \xi_t^{f,(J),*} B_t^r - \xi_t^{m,*} B_t^m = U_t^* = \tilde{U}_t^*(J)^*.
\]

The change in the value of the portfolio on this set is

\[
\sum_{i \in J^c} \xi_u^{i,*} dP_u^i + \xi_t^{f,(J),*} dP_t^i + \xi_t^{C,(J),*} dP_C^i + \xi_t^{f,(J),*} dB_t^r - \xi_t^{m,*} dB_t^m = \left( \sum_{i \in J^c} (r_D + h_i^Q) \left( \tilde{U}_t^{(i),*} \right) \right) + \left( L_I (\tilde{V}_t - M_{-})^+ + \tilde{U}_t^{(J),*} \right) \mu_I + \left( -L_C (\tilde{V}_t - M_{-})^- + \tilde{U}_t^{(J),*} \right) \mu_C + r_m (M_t) M_t + r_f (\xi_t^{f,(J),*}) \xi_t^{f,(J),*} B_t^r dt.
\]

Similarly to the case of a single name credit default swap, the super-replicating strategy needs to also include the cash flow

\[
\left( r_f (\xi_t^{f,(J),*}) - r_D \right) \sum_{i \in J^c} L_i dt,
\]

due to the fact that \(\tilde{V}\) is obtained by discounting at the rate \(r_D\), rather than \(r_f\), and hence the loss given default rates \(\sum_{i \in J^c} L_i dt\) accrues interest at rate \(r_D\).

The change in value of the super-replicating portfolio is obtained by using equations (63) and (64), and needs to be compared with the change in the valuation process, given by

\[
d\tilde{U}_t^{(J),*} = \left( (r_D + h_i^Q) (L_I (\tilde{V}_t - M_{-})^+ + \tilde{U}_t^{(J),*}) - (\mu_C)^* (\tilde{V}_t, M_t, \tilde{U}_t^{(J),*}) (L_C (\tilde{V}_t - M_{-})^- - \tilde{U}_t^{(J),*}) \right) + \sum_{k \in J^c} (r_D + h_i^Q) \tilde{U}_t^{(k) \cup (J),*} + r_m (M_t) M_t + r_f (\xi_t^{f,(J),*}) \left( (\xi_t^{f,(J),*}) B_t^r + \sum_{i \in J^c} L_i \right) - r_D \sum_{i \in J^c} L_i dt
\]

It then follows that (63) together with (64) dominates (65) from above because

\[
\left( (\mu_C)^* (\tilde{V}_t, M_t, \tilde{U}_t^{(J),*}) - \mu_C \right) (\tilde{\theta}_C (\tilde{V}_t, M_t) - \tilde{U}_t^{(J),*}) \geq 0.
\]

To complete the proof, it is left to consider the set \(\{t = \min_{k \in J^c} \tau_k^J \land \tau_C \land \tau_T \land T\}\), i.e., when \(t\) corresponds to a default time. Assume that the reference entity defaulting at \(t\) is \(k_0 \in J^c\). Then, by the definition of super-replicating strategy in (60), and specifically \(\xi_t^{k_0,(J),*}\), it follows that the value of the super-replicating portfolio drops from \(\tilde{U}_t^{(J),*}\) to \(\tilde{U}_t^{(k_0) \cup (J),*}\). By the induction hypothesis, \(\tilde{U}_t^{(k_0) \cup (J),*} \geq \tilde{U}_t^{(k) \cup (J),*}\). Together with Theorem 5.9, it follows that on the set \(\{\tau^J \land \tau_C \land \tau_T \land T < t \leq \min_{k \in J^c} \tau_k^J \land \tau_C \land \tau_T \land T\}\) the super-replicating portfolio dominates \(\tilde{U}_t^{(J)}\). By summing over the indicator sets as in (59)-(61), we get this dominance for all \(0 \leq t \leq \tau_C \land \tau_T \land T\).
We expect that, as the number of reference entities increases, so does the difference between sub-replication and super-replication valuations. This may be intuitively understood as follows. The terminal/closeout condition of the super-replication in the case of a single reference entity matches the terminal/closeout condition of the XVA. However, as shown in Theorem 5.10, in the case of a credit default swap portfolio where multiple reference entities appear, the terminal/closeout of the super-replication dominates that of the XVA. Thus, in the case of two reference entities, the terminal/closeout condition includes the jump to the closeout/terminal condition for the single reference entity case, in addition to the cash flows accumulated prior to default. It is thus expected that the difference between the super-replication and the XVA in the case of two reference entities is greater than the corresponding difference when the portfolio consists of a single CDS. Iterating this reasoning inductively, we conclude that the difference between the super-replication valuation and the XVA grows as more CDS contracts are added to the portfolio. This highlights the importance of the proposed robust approach, as opposed to the naive approach, which plugs one of the two extremes of the counterparty’s intensity uncertainty interval into the valuation formulas.

6 Conclusions

We have developed a framework to calculate the robust XVA price process of a credit default swap portfolio. We have focused on the situation where the trader faces uncertainty on the return rate of the bond underwritten by her counterparty. The credit default swap portfolio is replicated by the investor using defaultable bonds underwritten by the same entities referencing the single name credit default swap contracts in the portfolio. By constraining the return rate of the counterparty bond to lie within an uncertainty interval, we have derived lower and upper bounds for the XVA. Our analysis highlights a nontrivial interaction between the value process of the trade that accounts for all financing costs, and the closeout process that depends on the clean price of the transaction. The latter is obtained by pricing the cash flow of the trade, ignoring all other costs involved. We have shown that the value of the super-replication is obtained by switching, possibly multiple times during the life of the transaction, between the lower and upper bound of the bond rate uncertainty interval, depending on whether the value of the XVA replication trade lies above or below the corresponding close-out value of the transaction. The difference between the super-replication and the sub-replication valuation of the XVA process is expected to increase if the traded portfolio consists of a larger number of CDS contracts.

A Proofs of lemmas and propositions

Proof of Eq. (18). First, observe that the linear BSDE (17) admits the solution given by

\[ 
\hat{V}_t = \hat{C}_1(t) = -\mathbb{E}^{\mathbb{Q}}\left[ \int_t^{\tau_1 \wedge T} e^{-r_D(u-t)} S_1 \, du - L_1 e^{-r_D(\tau_1-t)} 1_{\{t \leq \tau_1 \}} | \mathcal{F}_t \right] 1_{\{t \leq \tau_1 \}}. 
\]
Moreover, as the default distribution is characterized by $\mathbb{Q}[u \geq \tau_1] = e^{-\int_0^u h_1^Q(s) \, ds}$, on the same event $\{t \leq \tau_1\}$ we have that

$$
\dot{V}_t = -\mathbb{E}^\mathbb{Q}\left[ \int_t^T \mathbb{1}_{\{u \geq \tau_1\}} e^{-r_P(u-t)} S_1 \, du - \int_t^T L_1 e^{-r_P(u-t)} h_1^Q(u) e^{-\int_t^u h_1^Q(s) \, ds} \, du \mid \mathcal{F}_t \right] \mathbb{1}_{\{t \leq \tau_1\}}
= -\mathbb{E}^\mathbb{Q}\left[ \int_t^T e^{-\int_t^u (h_1^Q(s)+r_P) \, ds} S_1 \, du - \int_t^T L_1 h_1^Q(u) e^{-\int_t^u (h_1^Q(s)+r_P) \, ds} \, du \mid \mathcal{F}_t \right] \mathbb{1}_{\{t \leq \tau_1\}}.
$$

(67)

**Proof of Proposition 4.6.** To facilitate the no-arbitrage argument, we will express the wealth process under a suitable measure $\mathbb{P}$ specified via the stochastic exponential

$$
d\mathbb{P}^i = \prod_{i \in \{1, \ldots, N, I, C\}} \left( \frac{\mu_i - r_f^i}{\int_0^{\tau_1} h_i^P(s) \, ds} \right)^H_i \exp \left( \int_0^{\tau_1} r_f^i \, dt - \mu_i + h_i^P(s) \, ds \right).
$$

By Assumption 4.4 this change of measure is well defined. Moreover, while the measure $\mathbb{P}$ is unknown to the investor, there is no issue with using it from an abstract point of view to rule out arbitrage. By Girsanov’s theorem, the dynamics of the risky assets are given by

$$
dP_i^t = r_f^i P_i^t \, dt - P_i^t \, d\mathbb{P}^i,
$$

(68)

for $i \in \{1, \ldots, N, I, C\}$ where $\mathbb{P}^i := (\mathbb{P}^i_t; 0 \leq t \leq \tau)$ are $(\mathbb{G}, \mathbb{P})$-martingales. The $r_f^i$ discounted assets $\tilde{P}_i^t := e^{-r_f^i t} P_i^t$ are thus $(\mathbb{G}, \mathbb{P})$-martingales. In particular, the default intensities under $\mathbb{P}$ are given by $h_i^P = \mu_i - r_f^i$, which are positive by Assumption 4.4.

Denote the wealth process associated with $(P_i^t; i \in \{1, \ldots, N, I, C\})_{t \geq 0}$ in the underlying market by $\tilde{V}_t$. Using the self-financing condition, its dynamics are given by

$$
d\tilde{V}_t = r_f \xi^t_i B_i^t \, dt + \sum_{i \in \{1, \ldots, N, I, C\}} r_f^i \xi^t_i dP_i^t
= \left( r_f \xi^t_i B_i^t + \sum_{i \in \{1, \ldots, N, I, C\}} r_f^i \xi^t_i P_i^t \right) dt - \sum_{i \in \{1, \ldots, N, I, C\}} \xi^t_i P_i^t \, d\mathbb{P}^i_t.
$$

(69)

Then we observe that $r_f \xi^t_i \leq r_f^i \xi^t_i$ and thus

$$
\tilde{V}_t(\varphi, x) - \tilde{V}_0(\varphi, x) = \int_0^T \left( r_f \xi^t_i B_i^t + \sum_{i \in \{1, \ldots, N, I, C\}} r_f^i \xi^t_i P_i^t \right) dt - \sum_{i \in \{1, \ldots, N, I, C\}} \int_0^T \xi^t_i P_i^t \, d\mathbb{P}^i_t
\leq \int_0^T \left( r_f \xi^t_i B_i^t + \sum_{i \in \{1, \ldots, N, I, C\}} r_f^i \xi^t_i P_i^t \right) dt - \sum_{i \in \{1, \ldots, N, I, C\}} \int_0^T \xi^t_i P_i^t \, d\mathbb{P}^i_t
= \int_0^T \left( r_f \tilde{V}_t(\varphi, x) + \sum_{i \in \{1, \ldots, N, I, C\}} \int_0^T r_f^i \xi^t_i dP_i^t \right) dt.
$$

Therefore, it follows that

$$
e^{-r_f^t \tau} \tilde{V}_t(\varphi, x) - \tilde{V}_0(\varphi, x) \leq \sum_{i \in \{1, \ldots, N, I, C\}} \int_0^T r_f^i \xi^t_i dP_i^t.
$$
Then, assuming, from which it follows that some interval inequality. Fortunately, by the classical Picard-Lindelöf Theorem the solution to eq. (22) exists on argument above that shows that we have that \( \lim_{t \to \infty} \), and therefore is a supermartingale. Taking expectations, we conclude that
\[
\mathbb{E}^\tilde{P} \left[ e^{-r^+_f \tau} \tilde{V}_\tau(\varphi, x) - \tilde{V}_0(\varphi, x) \right] \leq 0.
\]
Thus either \( \tilde{V}_\tau(\varphi, x) = e^{r^+_f \tau} x \) or \( \mathbb{P}[\tilde{V}_\tau(\varphi, x) < e^{r^+_f \tau} x] > 0 \). As \( \tilde{P} \) is equivalent to \( \mathbb{P} \), this shows that arbitrage opportunities for the investor are precluded in this model (he would receive \( e^{r^+_f \tau} x \) by lending the positive cash amount \( x \) to the treasury desk at the rate \( r^+_f \)).

**Proof of Proposition 5.3.** The existence and uniqueness of a solution to ODE (32) on the time interval \([0, T]\) follows from the classical Picard-Lindelöf Theorem, together with Corollary II.3.2 of Hartman (2001).

We now show existence and uniqueness of a solution to ODE (33). The existence again follows from the classical Picard-Lindelöf Theorem on every continuity interval of \( h_i \)'s. For simplicity of exposition we will assume that all \( h_i \)'s are continuous on \([0, T]\). In case, of a discontinuity, the solution will not be differentiable there, but will remain continuous.

First note that \( u_t \) is bounded. To see this, observe that \( \tilde{g} \) is Lipschitz in its second argument, and \( |\tilde{g}(t, 0; \tilde{v}(t), m(t))| \leq K_0 \) is uniformly bounded, by possibly increasing the constant \( K_0 \) if needed. It thus follows that
\[
|\tilde{g}(t, \tilde{u}; \tilde{v}(t), m(t))| \leq |\tilde{g}(t, \tilde{u}(t); \tilde{v}(t), m(t)) - \tilde{g}(t, 0; \tilde{v}(t), m(t))| + |\tilde{g}(t, 0; \tilde{v}(t), m(t))| \leq K_0 |\tilde{u}| + K_0.
\]
(70)

Then, assuming, \( \tilde{u} \) is differentiable, we can employ Gronwall inequality and deduce that if
\[
\partial_t \tilde{u}(t) \leq K_0 \tilde{u}(t) + K_0,
\]
(71)

then \( \tilde{u}(t) \leq K_1 := K_0 T e^{K_0 T} \), for \( t \in [0, T] \). Similar for the lower bound, if
\[
\partial_t \tilde{u}(t) \geq -K_0 \tilde{u}(t) - K_0
\]
(72)

from which it follows that \( \tilde{u}(t) \geq -K_1 \).

We would have been done, if not for the assumption of differentiability needed for the Gronwall inequality. Fortunately, by the classical Picard-Lindelöf Theorem the solution to eq. (22) exists on some interval \([0, T_0) \), \([-K_1 - 1, K_1 + 1] \) that is for \( t \in [0, T_0) \) it holds that \( |\tilde{u}(t)| \leq K_1 + 1 \), and it is unique there. This time, we are guaranteed differentiability. Assume by contradiction that it cannot be extended (to the right) beyond \( T_0 \) and that \( T_0 < T \) (the same argument applies, if \( T_0 = T \), but the solution cannot be extended to the closed interval \([0, T]\)). Then by Corollary II.3.2 of Hartman (2001) we have that \( \lim_{t \to T_0} |\tilde{u}(t)| = K_1 + 1 \). We now reach a contradiction, by employing Gronwall inequality argument above that shows that \( |\tilde{u}| \leq K_1 \).

\[\square\]
Proof of Proposition 5.1. As the filtration $\mathcal{F}$ is trivial, the $\mathcal{F}$-BSDE is in fact an ODE. The existence and uniqueness to this ODE is shown in Proposition 5.3. The equivalence of the full $\mathcal{G}$-BSDEs and the reduced $\mathcal{F}$-BSDEs follows from the projection result (Crépey and Song, 2015, Theorem 4.3) as condition (A) in their paper is satisfied by our assumptions on the filtrations and their Condition (J) is also satisfied (as the terminal condition does not depend on $\bar{Z}$, $\bar{Z}^l$ and $\bar{Z}^C$). Finally, by the martingale representation theorems with respect to $\mathcal{F}$ and $\mathcal{G}$ (see (Bielecki and Rutkowski, 2001, Section 5.2); their required assumptions are satisfied because our intensities are bounded), the solution of our BSDEs and those of the martingale problems considered in Crépey and Song (2015) coincide. 

Proof of Theorem 5.4. First, note that similar to the proof of Proposition 5.3, the functions $\tilde{u}^*$ and $\tilde{u}_*$, defined as the solutions to the ODEs in (34) exist and are unique. This follows from the fact that the functions $g^*$ and $g_*$ are Lipschitz continuous in all arguments.

Assume, by contradiction, that there exists $T_0 \leq T$ for which $\tilde{u}^*(T_0) < \tilde{u}(T_0)$, and set $T_1 = \sup \{ t \leq T_0 | \tilde{u}^*(t) \geq \tilde{u}(t) \}$. We have that $T_1$ is well defined, and $T_1 \geq 0$, because $\tilde{u}^*(0) = \tilde{u}(0) = 0$ and $\tilde{u}^*(t) < \tilde{u}(t)$ for $t \in (T_1, T_0)$. Using the facts that $\mu_{C} > r_D$ and that $(\mu_{C}^Q)^*(\tilde{v}, m, \tilde{u})(\tilde{C}(\tilde{v}, m) - \tilde{u}) \geq \mu_{C}^Q(t)(\tilde{C}(\tilde{v}, m) - \tilde{u})$ for any $t \in [0, T]$, we have that

$$\partial_t \tilde{u}^*(T_1) = g^*(T_1, \tilde{u}^*; \tilde{v}, m)$$

$$= h_{T}^G(\tilde{\theta}_T(\tilde{v}(T_1), m(T_1))) - h_{T}^Q \tilde{u}^*(T_1)$$

$$+ ((\mu_{C})^*(\tilde{v}(T_1), m(T_1), \tilde{u}^*(T_1)) - r_D)(\tilde{C}(\tilde{v}(T_1), m(T_1)) - \tilde{u}^*(T_1))$$

$$+ f(T_1, \tilde{u}^*(T_1), \tilde{\theta}_T(\tilde{v}(T_1), m(T_1))) - \tilde{u}^*(T_1)$$

$$\geq h_{T}^G(\tilde{\theta}_T(\tilde{v}(T_1), m(T_1))) - \tilde{u}^*(T_1) + h_{T}^Q(\tilde{C}(\tilde{v}(T_1), m(T_1)) - \tilde{u}^*(T_1)) - h_{T}^Q \tilde{u}^*(T_1)$$

$$+ f(T_1, \tilde{u}^*(T_1), \tilde{\theta}_T(\tilde{v}(T_1), m(T_1))) - \tilde{u}^*(T_1) - \tilde{\theta}_T(\tilde{v}(T_1), m(T_1))$$

$$= g(T_1, \tilde{u}; \tilde{v}(T_1), m(T_1)) dt = \partial_t \tilde{u}_*(T_1).$$

It follows that there exists an $\epsilon > 0$, such that $\tilde{u}^*(t) \geq \tilde{u}(t)$ for $t \in [T_1, T_1 + \epsilon]$. This contradicts the assumption, and proves the theorem.

Proof of Theorem 5.9. The proof of the super-replicating strategies is done by induction over $|J|$, i.e., the cardinality of $J$. Without loss of generality, we may assume $\gamma \in \{1, -1\}$. We present the proof for $\gamma = 1$, as this is identical to the case $\gamma = -1$. For notational convenience, we drop the superscript $\gamma$. If $|J| = N - 1$, the thesis follows directly from Theorem 5.5. By induction over the cardinality of $|J|$, assume that the result holds in case of when the entities in the set $J$ have not defaulted yet, with $|J| = n + 1 \geq 1$. Next, we prove the result for the case when the set $J$ of entities that have not defaulted yet has cardinality $n$. Fix such a set $J$ for which $|J| = n$. Assume, by contradiction, that there exists $T_0 \leq T$ for which $\tilde{u}(J)^*(T_0) < \tilde{u}(T_0)$, and set $T_1 = \sup \{ t \leq T_0 | \tilde{u}(J)^*(t) \geq \tilde{u}(t) \}$. Then, $T_1$ is well defined, and $T_1 \geq 0$ since $\tilde{u}(J)^*(0) = \tilde{u}(0) = 0$ and $\tilde{u}(J)^*(t) < \tilde{u}(t)$ for $t \in (T_1, T_0)$. Denote $Z^{f(J),*} = \sum_{k \in J^c} \tilde{u}(k)^{J,J^c} - (|J^c| + 1)\tilde{u}(J)^*(T_1) + \tilde{\theta}_J(\tilde{v}(T_1), m(T_1)) + \tilde{\theta}_C(\tilde{v}(T_1), m(T_1)) + \sum_{k \in J^c} L_k - M_t$, and similarly, $Z^{f(J)} = \sum_{k \in J^c} \tilde{u}(k)^{J,J^c} - (|J^c| + 1)\tilde{u}(J)^*(T_1) + \tilde{\theta}_J(\tilde{v}(T_1), m(T_1)) + \tilde{\theta}_C(\tilde{v}(T_1), m(T_1)) + \sum_{k \in J^c} L_k - M_t$.

Using the facts that $\mu_{C} > r_D$ and that $(\mu_{C}^Q)^*(\tilde{v}, m, \tilde{u})(\tilde{C}(\tilde{v}, m) - \tilde{u}) \geq \mu_{C}^Q(t)(\tilde{C}(\tilde{v}, m) - \tilde{u})$ for any
$t \in [0, T]$ we have that

$$
\partial_t \hat{u}^{(J),*} = g^* \left( T_1, \hat{u}^{(J),*}, \sum_{k \in J_c} \hat{u}^{(k) \cup J_c}, \sum_{k \in J_c} h_Q^* \hat{u}^{(k) \cup J_c}; \hat{u}^{(J),*}, m, J \right)
$$

(74)

$$
= h_Q^* \left( T_1 \right) \left( \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) + \sum_{i \in J_c} h_N^* \left( \tilde{u}^{(i) \cup J_c},*\left( T_1 \right) - \hat{u}^{(J),*}(T_1) \right)
$$

$$
+ (\mu C)^*(\hat{u}(T_1), m(T_1), \hat{u}^{(J),*}(T_1)) - r_D \left( \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right)
$$

$$
+ f(T_1, \hat{u}^{(J),*}(T_1), \sum_{k \in J_c} \hat{u}^{(k) \cup J_c} - |J_c| \hat{u}^{(J),*}(T_1), \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1)),
$$

$$
\tilde{\theta}_C(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}; \hat{u}(T_1), m(T_1), J)
$$

$$
= h_Q^* \left( T_1 \right) \left( \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) - \sum_{i \in J_c} h_N^* \hat{u}^{(J),*}(T_1)
$$

$$
+ (\mu C)^*(\hat{u}(T_1), m(T_1), \hat{u}^{(J),*}(T_1)) - r_D \left( \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) - r_f \left( Z^{(J),*} \right) Z^{(J),*}
$$

$$
+ \sum_{k \in J_c} \left( h_Q^* + r_D \right) \tilde{u}^{(i) \cup J_c}(T_1)
$$

$$
+ r_D \left( -(|J_c| + 2)\hat{u}^{(J),*}(T_1) + \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) \right) - r_m(M_t) M_t + r_D \sum_{k \in J_c} L_k
$$

$$
\geq h_Q^* \left( T_1 \right) \left( \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) - \sum_{i \in J_c} h_N^* \hat{u}^{(J),*}(T_1)
$$

$$
+ (\mu C)^*(\hat{u}(T_1), m(T_1), \hat{u}^{(J),*}(T_1)) - r_D \left( \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) - r_f \left( Z^{(J),*} \right) Z^{(J),*}
$$

$$
+ \sum_{k \in J_c} \left( h_Q^* + r_D \right) \tilde{u}^{(i) \cup J_c}(T_1)
$$

$$
+ r_D \left( -(|J_c| + 2)\hat{u}^{(J),*}(T_1) + \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) \right) - r_m(M_t) M_t + r_D \sum_{k \in J_c} L_k
$$

$$
= h_Q^* \left( T_1 \right) \left( \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right) - \sum_{i \in J_c} h_N^* \hat{u}^{(J),*}(T_1)
$$

$$
+ (\mu C)^*(\hat{u}(T_1), m(T_1), \hat{u}^{(J),*}(T_1)) - r_D \left( \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) - \hat{u}^{(J),*}(T_1) \right)
$$

$$
- r_f \left( Z^{(J),*} \right) \left( -(|J_c| + 1)\hat{u}^{(J),*}(T_1) + \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) + \sum_{k \in J_c} L_k - M_t \right)
$$

$$
+ \sum_{k \in J_c} \left( h_Q^* + r_D - r_f \left( Z^{(J),*} \right) \right) \tilde{u}^{(i) \cup J_c}(T_1)
$$

$$
+ r_D \left( -(|J_c| + 2)\hat{u}^{(J),*}(T_1) + \tilde{\theta}_J(\hat{u}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{u}(T_1), m(T_1)) \right) - r_m(M_t) M_t + r_D \sum_{k \in J_c} L_k
$$

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\begin{align*}
\geq h^Q_i(T_1)(\tilde{\theta}_I(\hat{v}(T_1), m(T_1)) - \tilde{u}^{(J),*}(T_1)) - \sum_{i \in J^c} h^Q_i C(\hat{v}(T_1), m(T_1)) + \sum_{k \in J^c} L_k - M_t \\
+ h^Q_C(\tilde{\theta}_C(\hat{v}(T_1), m(T_1)) - \tilde{u}^{(J),*}(T_1)) \\
- r_f(Z^{f,(J)})(-(|J^c| + 1)\tilde{u}^{(J),*}(T_1) + \tilde{\theta}_I(\hat{v}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{v}(T_1), m(T_1)) + \sum_{k \in J^c} L_k \\
+ \sum_{k \in J^c} \left(h^Q_k + r_D - r_f(Z^{f,(J)})\tilde{u}^{(f)\cup(J)}(T_1)\right) \\
+ r_D \left(-(|J^c| + 2)\tilde{u}^{(J),*}(T_1) + \tilde{\theta}_I(\hat{v}(T_1), m(T_1)) + \tilde{\theta}_C(\hat{v}(T_1), m(T_1))\right) - r_m(M_t)M_t + r_D \sum_{k \in J^c} L_k \\
= \hat{g}(T_1, \hat{u}; \hat{v}(T_1), m(T_1))dt = \partial_t \hat{u}(T_1).
\end{align*}

The first inequality above follows from the following inequality \( r_f(Z^{f,(J)})(-Z^{f,(J)}) \geq r_f(Z^{f,(J)},*)(Z^{f,(J)})^* \). To deduce the second inequality above, we have used that \( r_f^-, r_f^+ < \min_{i \in \{1, \ldots, N, I\}} \mu_i^L \mu_i^C \), and the induction hypothesis for sets of cardinality \( n + 1 \). This implies that there exists a constant \( \epsilon > 0 \), such that \( \tilde{u}^{*}(t) \geq \tilde{u}(t) \) for \( t \in [T_1, T_1 + \epsilon] \). This leads to a contradiction, and hence the theorem is proven.

\[ \square \]

References


J. Gregory. The XVA challenge, counterparty credit risk, funding, collateral, and capital, 3rd edition, 2015. John Wiley & Sons Ltd.


D. Van Deventer. Municipal credit default swap activity jumps, but overall market is still thin. Available at https://seekingalpha.com/article/2309325.