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Market Stability and Indifference Prices

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Based on
“Stability of Utility Maximization in Nonequivalent Markets,”
*Finance & Stochastics* (2016)
**Basic Problem**

Consider a financial market consisting of
- Bank account with $r = 0$
- Traded risky asset $S$
- Derivative security $\phi(S')$ with non-traded underlying $S'$ with

\[
\text{corr}(S', S) \approx 1
\]

**Question:**

\[
\text{corr}(S', S) \rightarrow 1 \quad ? \quad \text{price}(\phi(S')) \rightarrow \text{price}(\phi(S))?
\]

**Examples:**
- Derivatives on oil
- Hedging with an index
Related Work

- Exponential investor where $S, S'$ are correlated geometric Brownian motions:
  - Davis (working paper 2000) - value function
  - Monoyois (QF 2004) - indifference prices

- Utility Maximization Stability Literature:
  - Larsen-Žitković (SPA 2007)
  - Bayraktar-Kravitz (SPA 2013)
  - Kardaras-Žitković (MF 2011)

- Main hurdle: All previous stability work crucially relies on a fixed volatility structure.
Counterexample (I) – Set-up

Let $B$ and $W$ be two independent Brownian motions.

Financial Market:
- Bank account with $r = 0$
- Stock (indexed by $\rho \in (-1,1)$)

\[
dS^\rho_t = S^\rho_t \left( dt + \sqrt{1-\rho^2} dB_t + \rho dW_t \right), \quad S^\rho_0 := 1
\]

- Financial derivative with payoff $\phi(B_T)$:

\[
\phi : \mathbb{R} \to \mathbb{R} \text{ is bounded, continuous, not constant}
\]

Let $\phi_{\text{min}} := \inf_{x \in \mathbb{R}} \phi(x)$

- Unless $\rho = 0$, the payoff $\phi(B_T)$ cannot be replicated.
Consider an investor with

- \( x_0 > -\phi_{\text{min}} \): Initial wealth
- \( U(x) = \frac{x^p}{p} \): Utility function (\( p < 1, p = 0 \) corresponds to log)

Objective:

\[
u(x_0, \rho) := \sup_{H \in \mathcal{A}(\rho)} \mathbb{E} \left[ U \left( x_0 + \int_0^T H dS^\rho + \phi(B_T) \right) \right],
\]

where

\[
\mathcal{A}(\rho) := \{ H : \exists K = K(H), \int_0^T H dS^\rho \geq -K \ \forall t \}.
\]

Question: Does \( u(x_0, \rho) \) converge to \( u(x_0, 0) \) as \( \rho \to 0 \)?
Counterexample (III) – Admissibility Constraints

Let $H \in \mathcal{A}(\rho)$ such that

$$x_0 + \int_0^T H dS^\rho + \phi(B_T) \geq 0.$$ 

Two Cases:

- $\rho = 0$: We can replicate $\phi(B_T) = p + \int_0^T \Delta dS^0$

$$x_0 + \int_0^T H dS^0 + \phi(B_T) = x_0 + p + \int_0^T (H + \Delta) dS^0 \geq 0$$

- $\rho \neq 0$: Cannot replicate $\phi(B_T)$ by trading in $S^\rho$. It can be shown (see El Karoui & Quenez, 1995) that

$$x_0 + \int_0^T H dS^\rho + \phi_{\min} \geq 0$$

Yet $\phi_{\min} = \inf_x \phi(x) < p$. Therefore, the $\rho \neq 0$ markets are strictly more restrictive than the $\rho = 0$ market.
Counterexample (IV) – Results

**Theorem** (W. 2016) \( u(x_0, \rho) \) does not converge to \( u(x_0, 0) \) as \( \rho \to 0 \).

For \( x_0 > 0 \), consider the value function without random endowment

\[
w(x_0, \rho) := \sup_{H \in \mathcal{A}(\rho)} \mathbb{E} \left[ U \left( x_0 + \int_0^T H \mathrm{d}S^\rho \right) \right].
\]

We define the *utility indifference price* of \( \phi(B_T) \) by \( p = p(x_0, \rho) \) such that

\[ u(x_0, \rho) = w(x_0 + p, \rho). \]

For \( \rho = 0 \), \( p(x_0, 0) \) is the unique arbitrage-free price.

**Corollary** (W. 2016) Indifference prices do not converge:

\[
\lim_{\rho \to 0} \sup_{\rho \to 0} p(x_0, \rho) < p(x_0, 0).
\]
Positive Result (I)

Problems arise from the property that $U(x) = -\infty$ for $x < 0$.

- Real-line utility function: $U : \mathbb{R} \rightarrow \mathbb{R}$ (main example is $U(x) = -e^{-ax}$)
- In a Brownian framework, for $1 \leq n \leq \infty$, we consider

$$dS^n = dM^n + \lambda^n d\langle M^n \rangle, \quad S^n_0 \in \mathbb{R}.$$ 

Suppose $S^n \rightarrow S^\infty$ and $\int_0^\cdot \lambda^n dM^n \rightarrow \int_0^\cdot \lambda^\infty dM^\infty$ in the semimartingale topology.

- Financial derivative payoff: $V_T \in L^\infty$.

**Theorem** (W. 2016) Under appropriate integrability and nondegeneracy conditions, the value functions and indifference prices converge as $n \rightarrow \infty$. 
Positive Result (II)

**Main Difficulty:** Because of the changing volatility structure, small perturbations in the limiting market’s investment strategies are not consistent with strategies in the pre-limiting markets.

- **Primal Problem:** Suppose \((H \cdot S^\infty)\) is \(S^\infty\)-admissible.
  
  Is \((H \cdot S^n)\) \(S^n\)-admissible?
  Is \(H\) even \(S^n\)-integrable?

- **Dual Problem:** Consider a dual element \(\mathcal{E}(-\lambda^\infty \cdot M^\infty)^T \mathcal{E}(L)^T\) where
  \[\langle L, (\lambda^\infty \cdot M^\infty) \rangle \equiv 0.\]
  
  But \(\langle L, (\lambda^n \cdot M^n) \rangle = ?\)
THANK YOU!