On a mean field model for 1D thin film droplet coarsening

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Abstract

A thin liquid film coating a solid substrate is unstable and the late stage morphology is essentially quasiequilibrium droplets connected by an ultra thin film. Droplets exchange mass and coarsening occurs – the total number of droplets $N(t)$ decreases while the average size of droplets increases. It is predicted that $N(t)$ obeys a power law $N(t) \sim ct^{-2/5}$ in the 1D case. We study a mean field model proposed by Gratton and Witelski for the one-dimensional thin film coarsening and prove a universal one-sided bound on $N(t)$ that partially justifies the power law. A corresponding bound on the average drop size is also obtained. Then we study the relation of this model and the mean field model for Ostwald ripening and show that they are equivalent after a nonlinear rescaling of time. Finally we present some further estimates on the coarsening rates for the Ostwald ripening.

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1 Introduction

The dewetting instabilities of a thin liquid film coating a solid substrate cause rupture and complicated morphological changes in the early stage and ultimately the formation of complex nonlinear patterns in the late stage. The patterns are essentially fluid droplets connected by an ultra thin residual film. The evolution of the droplets exhibits a coarsening phenomena where we observe the decrease of the total number of droplets and an increase in the average droplet size and the average distance between droplets.

Mathematically, the thin film dynamics is described by the lubrication theory [18]. Denote by $h = h(x, t)$ the height of the thin film. The evolution of h satisfies the following equation

$$\partial_t h + \nabla \cdot \left(h^3 \nabla \left(\Delta h - U'(h)\right)\right) = 0, \tag{1.1}$$

where $U(h)$ is an intermolecular potential that includes both attractive and repulsive effects and the balance produces a single minimum h_{min} for h, which corresponds to the globally stable ultra thin film. The only mathematically necessary properties of the potential U are:

$$\begin{aligned}
&U(h) \text{ has a unique minimum } h_{min},\\
&U'(h) \text{ has a unique maximum, } p_{max}, \text{ at the maximum droplet height } h_{max},\\
&U'(h) = o(h^{-1}) \text{ and } U(h) \to 0, \text{ as } h \to \infty,\\
&\lim_{h \to 0} U(h) = \infty.
\end{aligned}$$

See e.g. [1] for a discussion about a class of physically reasonable choices of U. Also note that the system dissipates the free energy

$$E(t) := \int_\Omega \frac{1}{2} |\nabla h|^2 + U(h)\, dx, \tag{1.2}$$

where the first term of E is the surface energy and Ω is the fixed domain of solid substrate covered by the thin film.

The steady-state solutions for the 1D version of (1.1) with appropriate boundary conditions are single fluid droplet with parabolic shapes [1], see also [9]. In the late stage, the solution for (1.1) consists of quasi-stationary parabolic droplets and coarsening occurs under a long time scale, where we observe a decrease of the total number and an increase of the average size of droplets [9].

There are two possible mechanisms for this coarsening. The first one is collapsing, where smaller ones shrink and collapse while bigger ones grow. The other is collision, where droplets move around and collide to form bigger ones. In both situations the total mass of droplets are conserved if we ignore the mass of the ultra thin film. These two distinct mechanisms in general coexist and the dynamics is studied in [10] using asymptotic analysis methods.

For the 1D thin film coarsening dynamics, it is shown by heuristic arguments, and verified by numerical simulations, that under both mechanisms the number of droplets $N(t)$ decreases following a temporal power law $N(t) \sim ct^{-2/5}$. As a consequence, the average distance between drops and the average mass of drops grow as a temporal power law $t^{2/5}$. The average height and width of drops grow as $t^{1/5}$. Also the energy decays as a temporal power law $t^{-1/5}$.

It turns out that the rigorous analysis of the coarsening rates are challenging for the PDE model (1.1), partially because of the nonlinear mobility h^3. No mathematically rigorous work is available, even for the 1D case.

In this paper, we study a mean field model proposed by Gratton and Witelski [11] for the 1D thin film coarsening dynamics. Our main result is to establish rigorous bounds on the number of droplets $N(t)$ that partially justify the power law. Then we are able to obtain rigorous bounds on the average drop mass, on the average drop height and drop width, and the critical mass.

The critical mass – which is generally different from the average mass – is a quantity that divides the drops into two groups: drops with mass bigger than the critical mass grow and those with mass smaller than the critical mass shrink and collapse. The critical mass defines a scale for the masses and will be shown to grow as $t^{2/5}$.

The mean field model deals with the quasistatic dynamics in the late stage of coarsening and only consider the collapse mechanism. This model is a simplification of the ordinary differential equation (ODE) systems for the motion of masses and centers of quasiequilibrium drops [9], and is consistent with the 1D version of the thin film equation (1.1). Details about the mean field model will be presented later in Section 2. Here we give a brief description.

Assume that at time t, there are $N(t)$ droplets that are near equilibrium on the 1D thin film of length $\mathcal{L}$. Also assume that the droplet width is much smaller than the distance between drops and the motion of their centers are slow. Let $L_* := \mathcal{L}/N$ be the average distance between drops.

Let M_k be the mass of the k^{th} droplet and the total mass is conserved:

$$\mathcal{M} := \sum_{k=1}^{N} M_k = \text{const.} \tag{1.3}$$

The pressure of each droplet is $P_k \sim M_k^{-1/2}$. Then the change of mass M_k is determined by the pressure gradient from its neighbors M_{k-1} and M_{k+1}. Assume that the average neighbor pressures can be approximated by an effective mean field pressure $P_* = M_*^{-1/2}$, which is independent of k, and that the distance between drops can be approximated by the average distance L_*. The pressure gradient can now be approximated by $2(P_* - P_k)/L_*$, here we have a factor 2 since each drop has

2 neighbors. So we have an ODE system coupled through the mean field pressure $P_* = M_*^{-1/2}$,

$$\frac{dM_k}{dt} = \frac{2}{L_*}\left(M_*^{-1/2} - M_k^{-1/2}\right) \qquad \text{for all } k. \tag{1.4}$$

Note that the effective mean field pressure $M_*^{-1/2}$ is independent of k (but varies with t) and is mathematically determined by the conservation of total mass (1.3),

$$M_*^{-1/2} = \frac{1}{N}\sum_{k=1}^{N} M_k^{-1/2}, \tag{1.5}$$

or equivalently

$$M_* = N^2\left(\sum_{k=1}^{N} M_k^{-1/2}\right)^{-2}. \tag{1.6}$$

M_* is the critical mass – according to (1.4), drops with mass bigger than M_* grow and those with mass smaller than M_* shrink and collapse. This system (1.4) applies until when isolated singularities (collapse events) occur when some droplet has $M_k \to 0$ in finite time, then the system is restarted with the remaining drops after relabeling.

Note that since $\mathcal{M}$ and $\mathcal{L}$ are both conserved, the ratio $\rho = \mathcal{M}/\mathcal{L}$ is a fixed parameter. In addition, because $\mathcal{M}$ is the total area of droplets, ρ is the average height of the thin film. Replacing $\mathcal{L}$ by $\mathcal{M}/\rho$, (1.4) becomes an equation involving the average mass $\mathcal{M}/N$,

$$\frac{dM_k}{dt} = \frac{2\rho N(t)}{\mathcal{M}}\left(M_*^{-1/2} - M_k^{-1/2}\right). \tag{1.7}$$

The dynamics of the mean field model is still complex due to the arbitrary distribution of drop sizes. On the other hand, even though it is much simpler than the nonlinear equation (1.1), it captures many of the key features of the dynamics and has several advantages. First, the mean field model deals with the late stage quasistatic dynamics derived from (1.1) with nonlinear mobility h^3. Second, it singles out the collapse mechanism and avoids the more complicated collision mechanism. Third, it gives a concrete definition of the critical mass – we don't know if there is a counterpart in the nonlinear equation setting (1.1). Another feature is, it makes it possible to get direct estimates on the drop numbers and hence average distance between drops and average mass of drops – there is no easy way to count the drop number in the nonlinear equation setting, so far as we know.

Since the number of drops $N(t)$ is monotone decreasing ("staircasing") – it keeps constant and then jumps to a lower plateau only when collapse events happen, it is immediate that the power law $N \sim ct^{-2/5}$ has to be considered in an average sense.

It is possible for $N(t)$ to decrease slower than the power law – for example, there are nontrivial unstable equilibrium states, and the system will approach the ultimate steady state and settle down. We know of no heuristic reason to prevent $N(t)$ from decreasing faster than the power law, even in the average sense. However, we will prove that faster decrease rate is impossible – again, in the average sense.

In [12], Kohn and Otto introduced a method to study the lower bound of the energy for Cahn-Hilliard equations. Their results are power-law bounds for energy decay rates in a time-averaged sense. This method has been applied to treat many other models [2, 3, 4, 5, 6, 7, 8, 13, 14, 19]. Specifically, in [19] Otto, Rump and Slepčev study an equation similar to (1.1) but with a linear mobility

$$\partial_t h + \nabla \cdot (h \nabla (\Delta h - U'(h))) = 0. \tag{1.8}$$

The linear mobility allows them to rewrite the equation into a gradient flow with respect to the Wasserstein distance and apply the Kohn-Otto method to get an estimate on the energy decay rate, which in the 1D case obeys the same temporal power law for the equation with nonlinear mobility (1.1).

Our model (1.4) also dissipates a free energy $E(t)$ which is defined as

$$E(t) = \frac{\sum_{k=1}^{N} M_k^{1/2}}{\mathcal{M}}. \tag{1.9}$$

However, the Kohn-Otto framework can not be immediately applied to our model due to the fact that our model explicitly depends on the average mass of drops. Nevertheless, by considering an additional relation between the free energy and the average mass, it turns out that, rather than a bound on the energy, we are able to obtain a lower bound for the number of droplets $N(t)$.

Our first result is a one-sided time-averaged estimate on $N(t)$ which remains valid independent of initial conditions. It can also be interpreted as an estimate on the growth rate of the average mass of drops. The validity time T depends on the initial value of a dual quantity $S(t)$ defined by

$$S(t) = \frac{\sum_{k=1}^{N} M_k^{3/2}}{\mathcal{M}}, \tag{1.10}$$

which scales as square root of average mass.

Theorem 1.1. *There exist positive constants C_1 and C_2 independent of all system parameters such that for any solution $\{M_k\}$ of the mean field model (1.4), we have*

$$\int_0^T \left(\frac{N(t)}{\mathcal{M}} \right)^2 dt \geqslant C_1 \int_0^T (\rho t)^{-4/5} \, dt \qquad \text{if } \rho T \geqslant C_2 S(0)^5. \tag{1.11}$$

About the width and height of drops, it can be shown that they both are proportional to the square root of drop mass. The estimate for the average width and height is $\sum_{k=1}^{N} M_k^{1/2}/N \leqslant ct^{1/5}$. The estimate for the critical mass is $M_* \leqslant ct^{2/5}$. Both estimates are in average sense.

Corollary 1.2. *Let C_1 and C_2 be the same constants as in Theorem 1.1. Then for any solution $\{M_k\}$ of the mean field model (1.4), we have*

$$\int_0^T \left(\frac{\sum_{k=1}^{N} M_k^{1/2}}{N(t)} \right)^{-4} dt \geqslant C_1 \int_0^T (\rho t)^{-4/5} \; dt, \tag{1.12}$$

$$\int_0^T M_*(t)^{-2} \, dt \geqslant C_1 \int_0^T (\rho t)^{-4/5} \; dt \tag{1.13}$$

if $\rho T \geqslant C_2 S(0)^5$.

Our second result is, after a nonlinear rescaling of time, this model is equivalent to the classical Lifshitz-Slyozov-Wagner (LSW) mean field model for 2D Ostwald ripening.

Note that Ostwald ripening is the coarsening dynamics for the late stage of binary phase transition or phase separation in a dilute two-phase mixture, where drops of one phase dispersed in an environment of the other phase. The LSW mean field model for Ostwald ripening [15, 21] describes the evolution of the drop sizes and predicts a power law growth of average drop size. We will describe the model in Section 4. The energy decay rates for the LSW models for Ostwald ripening are studied in [5].

The structure of the paper is as follows. In Section 2, we will sketch the derivation of the Gratton-Witelski mean field model and give heuristic arguments about why we expect the power law $N \sim ct^{-2/5}$. In Section 3 we establish the necessary lemmas and prove our Theorem 1.1 and Corollary 1.2.

In Section 4, we discuss the relation of the Gratton-Witelski mean field model and the mean field model for 2D Ostwald ripening.

In Section 5, we will give some further estimates on the coarsening rates for mean field models for the Ostwald ripening, in any space dimensions. These are inspired by the study of the mean field model for thin film coarsening. The results include estimates on the average drop size, critical drop size, etc.

2 The Gratton-Witelski mean field model

2.1 Derivation of the mean field model

We briefly sketch the derivation of the mean field model. Interested readers are encouraged to read [11] for the detailed discussion and also [9, 10] for the profiles of steady state solutions.

We first consider the shape of steady state solutions for the 1D thin film equation

$$\partial_t h + \partial_x \left(h^3 \partial_x \left(\partial_{xx} h - U'(h) \right) \right) = 0. \tag{2.1}$$

The hydrodynamic pressure is given by

$$p := U'(h) - \partial_{xx} h \tag{2.2}$$

and the equilibrium solutions must have constant pressure $\bar{p}$, i.e.,

$$U'(h) - \partial_{xx} h \equiv \bar{p}. \tag{2.3}$$

In the 'core region' of a droplet where h is much bigger than the ultra thin film, $U'(h)$ is very small compared to the other two terms and hence the height has a parabolic profile, $\bar{h} \sim -\frac{\bar{p}}{2}(x^2 - \bar{w}^2)$ for $|x| < \bar{w}$, where $\bar{w}$ is the width of the droplet. The core region is connected to the surrounding ultra thin film and the width $\bar{w}$ is determined by the contact angle. It is described in [9] that the contact angle is determined by the intermolecular potential through a constant $A = A(U)$, which is the slope of the contact line and is independent of the droplet pressure $\bar{p}$. So $A \sim -\bar{h}'(\bar{w})$ and $\bar{w} \sim A/\bar{p}$. Now the mass, or area, of the droplet can be approximated by the integral

$$\bar{m} = \int_{-\bar{w}}^{\bar{w}} \bar{h}(x)\, dx \sim \frac{2A^3}{3\bar{p}^2}. \tag{2.4}$$

As a consequence of (2.4), the mass $\bar{m}$ completely characterizes the profile of a steady state droplet and the pressure $\bar{p} \sim \sqrt{2A^3/3}\bar{m}^{-1/2}$. And the drop width and height are

$$\bar{w} \sim \frac{A}{\bar{p}} \sim \sqrt{\frac{3}{2A}}\bar{m}^{1/2} \tag{2.5}$$

$$\bar{h}_{\max} \sim \frac{1}{2}\bar{p}\bar{w}^2 \sim \sqrt{\frac{3A}{8}}\bar{m}^{1/2} \tag{2.6}$$

In terms of energy, the droplet energy is now essentially determined by surface tension and hence

$$E \sim \frac{1}{2}\int_{-\bar{w}}^{\bar{w}} \left|\bar{h}'(x)\right|^2\, dx \sim \frac{1}{2}\left(\frac{2A^3}{3}\bar{m}\right)^{1/2} \tag{2.7}$$

Now we consider the dynamics of quasistatic drops in the late stage of coarsening. In [9], by studying equation (2.1), the asymptotic dynamics for quasistatic drops is described by ODEs for their pressures and positions. The ODE system for pressures can be translated into an ODE system for masses by the relation (2.4). Let M_{k-1}, M_k, M_{k+1} be masses of neighbor drops and L_k, L_{k+1} be the distances from drops $k-1$ to k and from k to $k+1$, respectively. Under a linear rescaling of time, the system reads

$$\frac{dM_k}{dt} = \frac{M_{k+1}^{-1/2} - M_k^{-1/2}}{L_{k+1}} - \frac{M_k^{-1/2} - M_{k-1}^{-1/2}}{L_k}. \tag{2.8}$$

Equation (2.8) can also be interpreted as that the change of mass of each droplet is determined by the pressure gradients of neighbor droplets. Since we assume the widths of droplets are much smaller than the distances between droplets, the pressure gradient from drops $k+1$ to k is

$$\frac{P_{k+1} - P_k}{L_{k+1}} \sim \sqrt{\frac{2A^3}{3}} \frac{M_{k+1}^{-1/2} - M_k^{-1/2}}{L_{k+1}}. \tag{2.9}$$

Assuming constant mobility, the change of M_k is then (2.8). Note that the constant mobility assumption is in fact not a coincidence – as was mentioned above [9], it is a result of (2.1).

Since drops are far away from each other, we can assume $L_{k+1} \sim L_k \sim L_* = \mathcal{L}/N$. Also if we assume that the average neighbor pressure $(M_{k+1}^{-1/2} + M_{k-1}^{-1/2})/2$ can be replaced by a global mean field pressure $M_*^{-1/2}$, then (2.8) can be rewritten as

$$\frac{dM_k}{dt} = \frac{2}{L_*} \left(M_*^{-1/2} - M_k^{-1/2} \right). \tag{2.10}$$

This is the Gratton-Witelski discrete mean field model. Here $L_* = \mathcal{L}/N$ is the average distance between droplets, and $M_*^{-1/2}$ is the mean field pressure, and is mathematically determined by the conservation of total mass (1.6).

Assuming further that the coarsening parameter (for details see [10, 11]) is small, the motion of droplet centers are small so that collision will not happen, or droplets are effectively fixed in space. Then (2.10) gives a complete simplified description of the coarsening dynamics. Isolated singularities (collapse events) occur when some droplet has $M_k \to 0$ in finite time.

The number $N(t)$ of droplets and the average distance $L_*(t)$ are piecewise constant – they change only when droplets collapse, which results in a decrease in $N(t)$ and increase in L_*.

Passing to the limit of a continuum distribution of drop masses $\phi(m,t)$, the total number and total mass of drops are

$$N(t) = \int_0^\infty \phi(m,t)\, dm, \qquad \mathcal{M} = \int_0^\infty m\phi(m,t)\, dm. \tag{2.11}$$

Note $N(t)$ is changing in time while the total mass $\mathcal{M}$ is conserved, i.e., we require the first moment of ϕ to be conserved.

Gratton and Witelski then consider a corresponding Lifshitz-Slyozov-Wagner (LSW) mean field model, which is a transport equation for ϕ.

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial m}\left(u(m)\phi\right) = 0, \tag{2.12}$$

where $u(m)$ is the flux from the discrete mean field model

$$u(m) = \frac{dm}{dt} = \frac{2}{L_*}\left(m_*^{-1/2} - m^{-1/2}\right) \tag{2.13}$$

and

$$m_*^{-1/2} = \frac{1}{N}\int_0^\infty m^{-1/2}\phi(m,t)\, dm, \qquad L_* = \frac{\mathcal{L}}{N}. \tag{2.14}$$

Now the discrete collapse events are statistically averaged out and all m_*, L_* and N are evolving continuously in time.

From here the self-similar solutions are studied and the long time behavior of $N(t)$ is shown by heuristic arguments and verified by numerical simulations to be $O(t^{-2/5})$, which is consistent with the expected thin film coarsening dynamics.

2.2 Heuristics of the power law $N \sim ct^{-2/5}$

Now we give a heuristic argument for the Gratton-Witelski mean field model (1.4) for 1D thin film coarsening to show why the droplet number $N(t)$ follows a temporal power law $N(t) \sim ct^{-2/5}$.

Recall that the mean field model is

$$\frac{dM_k}{dt} = \frac{2\rho N(t)}{\mathcal{M}}\left(M_*^{-1/2} - M_k^{-1/2}\right), \tag{2.15}$$

We consider the scaling of this model. Choose $\hat{m}$ and $\hat{t}$ as typical mass and time scales. Then $M_k = \hat{m}\tilde{M}_k, t = \hat{t}\tilde{t}, \mathcal{M} = \hat{m}\tilde{\mathcal{M}}$ and (2.15) becomes

$$\frac{\hat{m}^{5/2}}{\rho\hat{t}}\frac{d\tilde{M}_k}{d\tilde{t}} = \frac{2N(t)}{\tilde{\mathcal{M}}}\left(\tilde{M}_*^{-1/2} - \tilde{M}_k^{-1/2}\right), \tag{2.16}$$

This scaling indicates that $\hat{m}^{5/2}/\rho\hat{t}$ is a dimensionless constant. In other words, we expect a relation $\hat{m}^{5/2} \sim c_1\rho\hat{t}$. If we take the average mass $\mathcal{M}/N(t)$ as the typical mass, then we expect

$$\left(\frac{\mathcal{M}}{N(t)}\right)^{5/2} \sim c_1\rho t, \tag{2.17}$$

or

$$\frac{N(t)}{\mathcal{M}} \sim c_1^{-2/5}\,(\rho t)^{-2/5}\,. \tag{2.18}$$

As is discussed in section 1, we can only expect universal lower bounds on the drop number $N(t)$, and only in average sense.

3 Estimate on the coarsening rates

Recall that

$$\frac{dM_k}{dt} = \frac{2\rho N(t)}{\mathcal{M}}\left(M_*^{-1/2} - M_k^{-1/2}\right), \tag{3.1}$$

where $\mathcal{M}$ is the total mass (or area) of the droplets, and M_* is the critical mass determined by the conservation of $\mathcal{M}$, see (1.6).

We mention that the critical mass M_* is not the average mass. There is a simple relation between them.

Lemma 3.1. *(Critical mass is smaller than the average mass)*

$$M_* \leqslant \frac{\mathcal{M}}{N}. \tag{3.2}$$

Proof. By the Cauchy-Schwarz inequality,

$$\begin{aligned} N^2 &\leqslant \left(\sum_{k=1}^N M_k^{-1/2}\right)\left(\sum_{k=1}^N M_k^{1/2}\right) \\ &\leqslant \left(\sum_{k=1}^N M_k^{-1/2}\right)\left(N\sum_{k=1}^N M_k\right)^{1/2} = \left(\sum_{k=1}^N M_k^{-1/2}\right)(N\mathcal{M})^{1/2}\,. \end{aligned} \tag{3.3}$$

Hence by (1.6),

$$M_*^{1/2} = \frac{N}{\sum_{k=1}^N M_k^{-1/2}} \leqslant \left(\frac{\mathcal{M}}{N}\right)^{1/2}. \tag{3.4}$$

□

The relation of the average drop width/height and the average mass is the following lemma.

Lemma 3.2. *For any positive sequence* $\{M_k : k = 1, \cdots, N\}$, *if* $\mathcal{M} = \sum_{k=1}^N M_k$, *then we have*

$$\frac{\sum_{k=1}^N M_k^{1/2}}{N} \leqslant \left(\frac{\mathcal{M}}{N}\right)^{1/2}. \tag{3.5}$$

The proof is again a simply application of the Cauchy-Schwarz inequality and we omit the detail.

Remark. Estimates (1.12) and (1.13) in Corollary 1.2 are immediate consequences of Lemmas 3.2 and 3.1, respectively, if Theorem 1.1 is true. The remainder of this section will be devoted to the proof of Theorem 1.1, by considering the energy dissipation properties of the mean field model (3.1).

Similar to the full nonlinear thin film equation, the mean field model dissipates the free energy, which is now determined by the surface tension and whose average (with respect to total mass) is defined as

$$E(t) := \frac{\sum_{k=1}^N M_k^{1/2}}{\mathcal{M}}. \tag{3.6}$$

$E(t)$ is decreasing in time since

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2\mathcal{M}} \sum_{k=1}^N M_k^{-1/2} \frac{dM_k}{dt} = \frac{1}{2\mathcal{M}} \sum_{k=1}^N \left(M_k^{-1/2} - M_*^{-1/2}\right) \frac{dM_k}{dt} \\ &= -\frac{1}{4\rho N} \sum_{k=1}^N \left(\frac{dM_k}{dt}\right)^2 \leqslant 0.\end{aligned} \tag{3.7}$$

In (3.7) we used the fact that M_* is a constant and $\sum_{k=1}^N dM_k/dt = 0$.

The relation between $E(t)$ and the average mass $\mathcal{M}/N$ is given as follows.

Lemma 3.3. *For all* $t > 0$ *we have*

$$E(t) \leqslant \left(\frac{N(t)}{\mathcal{M}}\right)^{1/2}. \tag{3.8}$$

Proof. By Cauchy-Schwarz inequality,

$$E(t) = \frac{\sum_{k=1}^N M_k^{1/2}}{\mathcal{M}} \leqslant \frac{(N(t))^{1/2} \left(\sum_{k=1}^N M_k\right)^{1/2}}{\mathcal{M}} = \left(\frac{N(t)}{\mathcal{M}}\right)^{1/2}. \tag{3.9}$$

□

Equation (3.8) makes it possible to obtain estimates on the average mass by studying the energy decay rates. For our specific case, we need to generalize the Kohn-Otto framework [12]. Their framework involves an auxiliary dual quantity and we define ours as

$$S(t) := \frac{\sum_{k=1}^{N} M_k^{3/2}}{\mathcal{M}}. \tag{3.10}$$

Note that the definitions for E and S are in the same way as those in [5] and they both are continuous and piecewise differentiable.

The dual property between $E(t)$ and $S(t)$ is exhibited in Lemma 3.4 and the relation between their dissipation rates is in Lemma 3.5.

Lemma 3.4. *(Interpolation inequality)*

$$E(t)S(t) \geqslant 1 \qquad \text{for all } t. \tag{3.11}$$

Proof. This is an application of the Cauchy-Schwarz inequality.

$$\mathcal{M} = \sum_{k=1}^{N} M_k = \sum_{k=1}^{N} M_k^{1/4} M_k^{3/4} \leqslant \left(\sum_{k=1}^{N} M_k^{1/2} \sum_{k=1}^{N} M_k^{3/2} \right)^{1/2}. \tag{3.12}$$

So

$$E(t)S(t) = \frac{\sum_{k=1}^{N} M_k^{1/2} \sum_{k=1}^{N} M_k^{3/2}}{\mathcal{M}^2} \geqslant 1. \tag{3.13}$$

□

Lemma 3.5. *(Dissipation inequality) For all t when there is no droplet collapsing, we have*

$$\left(\frac{dS}{dt}\right)^2 \leqslant \frac{9\rho N(t)}{\mathcal{M}} \left(-\frac{dE}{dt}\right). \tag{3.14}$$

Proof. A direct calculation gives

$$\begin{aligned} \left|\frac{dS}{dt}\right| &= \frac{3}{2\mathcal{M}} \left| \sum_{k=1}^{N} M_k^{1/2} \frac{dM_k}{dt} \right| \\ &\leqslant \frac{3}{2\mathcal{M}} \left(\sum_{k=1}^{N} M_k \sum_{k=1}^{N} \left(\frac{dM_k}{dt}\right)^2 \right)^{1/2} \\ &= 3 \left(-\frac{\rho N(t)}{\mathcal{M}} \frac{dE}{dt} \right)^{1/2}. \end{aligned} \tag{3.15}$$

□

Now we are ready to present an estimate for $N(t)$ and $E(t)$.

Theorem 3.6. *There exist positive constants C_1 and C_2 independent of any system parameters such that*

$$\int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2 \, dt \geqslant C_1 \int_0^T (\rho t)^{-4/5} \, dt \quad \text{if } \rho T \geqslant C_2 S(0)^5. \tag{3.16}$$

Proof. The proof is based on the techniques developed in [12] and [5] and takes advantage of the additional relation (3.8) between $E(t)$ and $N(t)$.

Since $E(t)$ is a decreasing function of t, it is invertible and we can write t as a function of E. Then by (3.14)

$$\frac{9\rho N(t)}{\mathcal{M}} \left(-\frac{dE}{dt}\right) \geqslant \left(\frac{dS}{dt}\right)^2 = \left(\frac{dS}{dE}\frac{dE}{dt}\right)^2. \tag{3.17}$$

So

$$\frac{9\rho N(t)}{\mathcal{M}} \geqslant \left(\frac{dS}{dE}\right)^2 \left(-\frac{dE}{dt}\right). \tag{3.18}$$

Multiplying the above equation by $E(t)^2$ and integrating from 0 to T, we have

$$9\rho \int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2 \, dt \geqslant \int_{E_T}^{E_0} \left(\frac{dS}{dE}\right)^2 E^2 \, dE, \tag{3.19}$$

here $E_0 = E(0)$ and $E_T = E(T)$. Changing variable $\omega = E^{-1}$, we get

$$9\rho \int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2 \, dt \geqslant \int_{\omega_0}^{\omega_T} \left(\frac{dS}{d\omega}\right)^2 \, d\omega \geqslant \frac{(S_T - S_0)^2}{\omega_T - \omega_0}. \tag{3.20}$$

Here $S_T = S(T), S_0 = S(0)$ and $\omega_T = E_T^{-1}, \omega_0 = E_0^{-1}$.

(a.) If $S_T \geqslant 2S_0$, then

$$\begin{aligned} 9\rho \int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2 \, dt &\geqslant \frac{1}{4} S_T^2 \omega_T^{-1} = \frac{1}{4}(S_T E_T)^2 E_T^{-1} \\ &\geqslant \frac{1}{4} E_T^{-1} \quad (\text{ by } (3.11)) \\ &\geqslant \frac{1}{4} E_T^{-1/2} \left(\frac{N(T)}{\mathcal{M}}\right)^{-1/4} \quad (\text{ by } (3.8)). \end{aligned} \tag{3.21}$$

Define

$$h(T) := \int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2 \, dt. \tag{3.22}$$

Then

$$h'(T) = \frac{N(T)}{\mathcal{M}} E(T)^2 \tag{3.23}$$

and (3.21) indicates

$$h'(T)\,(36\rho h(T))^4 \geqslant 1. \tag{3.24}$$

(b.) Now if $S_T \leqslant 2S_0$, then by (3.8) and (3.11),

$$h'(T) = \frac{N(T)}{\mathcal{M}} E(T)^2 \geqslant E_T^4 \geqslant S_T^{-4} \geqslant (2S_0)^{-4}, \tag{3.25}$$

and consequently

$$h'(T)\,(2S_0)^4 \geqslant 1. \tag{3.26}$$

Combining (a.) and (b.), we see that for all $T > 0$

$$h'(T)\rho^4 \left(36h(T) + \frac{2}{\rho} S_0\right)^4 \geqslant 1. \tag{3.27}$$

Integrating the above inequality from 0 to T, we obtain

$$\frac{\rho^4}{180}\left(36h(T) + \frac{2}{\rho}S_0\right)^5 \geqslant \frac{\rho^4}{180}\left(\frac{2}{\rho}S_0\right)^5 + T \geqslant T. \tag{3.28}$$

And

$$36h(T) \geqslant \left(\frac{180}{\rho^4}T\right)^{1/5} - \frac{2}{\rho}S_0 \tag{3.29}$$

$$\geqslant \frac{1}{2}\left(\frac{180}{\rho^4}T\right)^{1/5} \quad \text{if} \quad \left(\frac{180}{\rho^4}T\right)^{1/5} > \frac{4}{\rho}S_0. \tag{3.30}$$

Hence

$$h(T) \geqslant \frac{1}{2}(180)^{-4/5}\int_0^T (\rho t)^{-4/5} \quad \text{if } \rho T \geqslant \frac{256}{45}S_0^5, \tag{3.31}$$

i.e.,

$$\int_0^T \frac{N(t)}{\mathcal{M}} E(t)^2\, dt \geqslant C_1 \int_0^T (\rho t)^{-4/5}\ dt \quad \text{if } \rho T \geqslant C_2 S_0^5 \tag{3.32}$$

with $C_1 = \frac{1}{2}(180)^{-4/5}$, and $C_2 = \frac{256}{45}$. □

Remark. Combining Lemma 3.3 and Theorem 3.6, we immediately obtain

$$\int_0^T \left(\frac{N(t)}{\mathcal{M}}\right)^2 dt \geqslant C_1 \int_0^T (\rho t)^{-4/5}\ dt \quad \text{if } \rho T \geqslant C_2 S(0)^5, \tag{3.33}$$

which is our Theorem 1.1.

4 Relation to the LSW model for 2D Ostwald ripening

Aside from the coarsening phenomena in thin liquid film, another important and widely observed coarsening phenomena is Ostwald ripening [20], which is related to the late stage of phase transition or phase separation in a dilute two-phase mixture, where droplets of one phase is widely spread in a matrix of another one. Bigger drops grow and smaller ones shrink and disappear in finite time, conserving the total mass of drops.

The Lifshitz-Slyozov-Wagner mean field theory [15, 21] was originally derived to describe Ostwald ripening. The discrete version is as follows [20, 16, 17]. Let n be the dimension of space. Suppose at time t there are $N(t)$ spherical drops of radius $\{R_k : k = 1, 2, \cdots, N\}$. The dynamics is determined by

$$\frac{dR_k}{dt} = \frac{1}{R_k}\left(\frac{1}{R_*} - \frac{1}{R_k}\right), \tag{4.1}$$

where $1/R_*$ is the mean field which is a spatial constant determined by the conservation of total mass $\sum_{k=1}^{N} R_k^n = \text{const.}$, or $\sum_{k=1}^{N} R_k^{n-1} dR_k/dt = 0$, which immediately gives

$$R_* = \frac{\sum_{k=1}^{N} R_k^{n-2}}{\sum_{k=1}^{N} R_k^{n-3}}. \tag{4.2}$$

R_* is the critical radius – according to (4.1), drops with radius bigger than R_* grow and those with radius smaller than R_* shrink and disappear. R_* is the average radius when $n = 3$ and the harmonic mean radius when $n = 2$.

Let $u_k = R_k^n$ be the rescaled mass of the k^{th} drop. Its evolution obeys

$$\frac{du_k}{dt} = n u_k^{(n-2)/n}\left(u_*^{-1/n} - u_k^{-1/n}\right), \tag{4.3}$$

where $u_* = R_*^n$.

Note that when $n = 2$, (4.3) becomes

$$\frac{du_k}{dt} = 2\left(u_*^{-1/2} - u_k^{-1/2}\right), \tag{4.4}$$

which is very similar to the Gratton-Witelski mean field model (3.1) for 1D thin film coarsening, the latter has just one extra factor on the right hand side. In fact we will show that they are equivalent.

For the Gratton-Witelski mean field model, define a new time scale

$$\tau = \int_0^t \frac{\rho N(s)}{\mathcal{M}}\, ds, \tag{4.5}$$

which is a strictly increasing, continuous and piecewise linear rescaling of t. Consequently t is a strictly increasing function of τ, $t = t(\tau)$.

In terms of the new time scale τ, if we define

$$\tilde{M}_k(\tau) := M_k(t(\tau)), \tag{4.6}$$

then (3.1) becomes

$$\frac{d\tilde{M}_k}{d\tau} = \frac{dM_k}{dt}\frac{dt}{d\tau} = 2\left(\tilde{M}_*^{-1/2} - \tilde{M}_k^{-1/2}\right) \tag{4.7}$$

and it is exactly the discrete classical LSW model for 2D Ostwald ripening (4.4).

The energy decay rates for the LSW mean field model for Ostwald ripening (4.3) were studied in [5]. We can immediately rewrite the results in [5] into an estimate for the Gratton-Witelski mean field model (4.7) for 1D thin film coarsening in the new time scale τ. Note that the energy E (3.6) and the dual quantity S (3.10) become $\tilde{E}(\tau) = E(t(\tau))$ and $\tilde{S}(\tau) = S(t(\tau))$.

Theorem 4.1. *There exist positive constants $\hat{C}_1$ and $\hat{C}_2$ such that*

$$\int_0^{\hat{\tau}} \tilde{E}(\tau)^2\, d\tau \geqslant \hat{C}_1 \int_0^{\hat{\tau}} \left(\tau^{-1/3}\right)^2\, d\tau \text{ for all } \hat{\tau} \geqslant \hat{C}_2\tilde{S}(0)^3. \tag{4.8}$$

We mention that the equivalence of the Gratton-Witelski discrete mean field model (3.1) and the LSW model for 2D Ostwald ripening (4.3) can be translated into their corresponding continuum versions, i.e., the transport equations for their distribution functions. Hence it is possible to apply known results for Ostwald ripening [16, 17] to the thin film coarsening model (2.12). Specifically explicit self-similar solutions for (2.12) can be obtained from those for the LSW model for 2D Ostwald ripening.

As to the wellposedness of the transport equation, unfortunately no result is known for the specific 2D LSW model. In [17], the wellposedness of a class of LSW transport equations is established, but this 2D case is not included. The reason is that in the 2D case, the critical radius R_* is the harmonic mean of the radii $\{R_k\}$ and it becomes zero when some drops disappear, causing the mean field $1/R_*$ to blow up. New techniques are needed to handle this situation.

Now we give some heuristic argument for the decay rate for E in the original time scale t.

Since we have proved in Theorem 1.1, in a weak sense, that $N/\mathcal{M} \geqslant ct^{-2/5}$, heuristically, assuming $\hat{M}$ to be the typical droplet mass, then $\hat{M} \sim \mathcal{M}/N$ and

$$E(t) \sim \hat{M}^{-1/2} \sim \left(\frac{N}{\mathcal{M}}\right)^{1/2} \geqslant ct^{-1/5}. \tag{4.9}$$

One expects to obtain an averaged lower bound for the decay rate of $E(t)$ in the time scale t. However, this seems unlikely for the Gratton-Witelski model, the reason being that the model (3.1) explicitly involves the average mass $\mathcal{M}/N$, which is related to the energy E through (3.8), but which is just a one-sided inequality.

The estimate (4.8) can not be translated into an estimate that involves just $E(t)$ and t. Nevertheless, it is consistent with the scaling (4.9). This is because *if* $N(t) \sim ct^{-2/5}$, then

$$\tau \sim \frac{c\rho}{\mathcal{M}} \int_0^t s^{-2/5}\, ds = \frac{5c\rho}{3\mathcal{M}} t^{3/5} \tag{4.10}$$

and *if* $\tilde{E}(\tau) \sim c_1 \tau^{-1/3}$ then

$$E(t) = \tilde{E}(\tau) \sim c_1 \tau^{-1/3} \sim c_2 t^{-1/5} \quad \text{with } c_2 = c_1 \left(\frac{5c\rho}{3\mathcal{M}}\right)^{-1/3}. \tag{4.11}$$

5 Further results on the coarsening rates for the LSW model for Ostwald ripening

In Section 4, we mention that the energy decay rates for the LSW mean field model for Ostwald ripening (4.1) and (4.2) were studied in [5]. The original LSW theory studied the distribution of radius and argued that the distribution will always approach a universal self-similar solution and the critical radius R_* should grow like a temporal power law

$$R_*(t) \sim ct^{1/3}. \tag{5.1}$$

Recall that R_* is defined as (4.2), i.e.,

$$R_* = \frac{\sum_{k=1}^N R_k^{n-2}}{\sum_{k=1}^N R_k^{n-3}}. \tag{5.2}$$

It has been proved in [16] that the distribution function may not always approach the universal self-similar solution – the limit of the distribution sensitively depends on the initial distribution in a subtle way.

The estimate for energy decay in [5] heuristically indicate that the typical radius will not grow faster than the temporal power law $t^{1/3}$ in an averaged sense.

Inspired by the study of the mean field model for 1D thin film coarsening, we come back to the Ostwald ripening case and want to show that the critical radius R_*, as well as the average radius, can not grow faster than the $t^{1/3}$ power law in an average sense, for any space dimension n.

Recall that when $\{R_k : k = 1, \cdots, N(t)\}$ are the radius of the spheres at time t, the average energy and dual quantity for the mean field model for Ostwald ripening are defined as ([5])

$$E_{Ost}(t) := \frac{\sum_{k=1}^N R_k^{n-1}}{\sum_{k=1}^N R_k^n}, \qquad S_{Ost}(t) := \frac{\sum_{k=1}^N R_k^{n+1}}{\sum_{k=1}^N R_k^n}. \tag{5.3}$$

The energy estimate in [5] says there exist positive constants C_1 and C_2 depending only on the space dimension n such that

$$\int_0^T E_{Ost}(t)^2 \, dt \geqslant C_1 \int_0^T t^{-2/3} \, dt \quad \text{for all } T \geqslant C_2 S_{Ost}(0)^3. \tag{5.4}$$

Now we consider the relations between the critical radius, the average radius, the average mass, and the average energy E_{Ost}. Note that the average radius is $\sum_{k=1}^N R_k/N$ and the average mass is $\sum_{k=1}^N R_k^n/N$.

Lemma 5.1. *For any positive sequence $\{R_k : k = 1, \cdots, N\}$, we have*

(a.) If $n < 3$, then the critical radius is smaller than the average radius; if $n = 3$, they are equal; if $n > 3$, the critical radius is bigger than the average radius, i.e.,

$$R_* \leqslant \frac{\sum_{k=1}^N R_k}{N} \quad \text{if } n < 3; \tag{5.5}$$

$$R_* = \frac{\sum_{k=1}^N R_k}{N} \quad \text{if } n = 3; \tag{5.6}$$

$$R_* \geqslant \frac{\sum_{k=1}^N R_k}{N} \quad \text{if } n > 3. \tag{5.7}$$

(b.) Both the critical radius and the average radius are smaller than the inverse of the average energy,

$$\frac{\sum_{k=1}^N R_k}{N} \leqslant \frac{1}{E_{Ost}}, \qquad R_* \leqslant \frac{1}{E_{Ost}}. \tag{5.8}$$

(c.) The average mass is smaller than the n^{th} *power of the inverse of the average energy,*

$$\frac{\sum_{k=1}^{N} R_k^n}{N} \leqslant \left(\frac{1}{E_{Ost}}\right)^n. \tag{5.9}$$

Proof. All statements follow from the Hölder inequality. The proof does not require n to be an integer. Recall that the Hölder inequality can be written as the following interpolation inequality.

For any positive sequence $\{f_k, g_k : k = 1, \cdots, N\}$ and positive number $0 < \alpha < 1$, we have

$$\sum_{k=1}^{N} f_k^{\alpha} g_k^{1-\alpha} \leqslant \left(\sum_{k=1}^{N} f_k\right)^{\alpha} \left(\sum_{k=1}^{N} g_k\right)^{1-\alpha} \tag{5.10}$$

(a.) If $n < 3$, choosing $\alpha = 1/(4-n)$, we have

$$\sum_{k=1}^{N} R_k^{n-2} = \sum_{k=1}^{N} R_k^{\alpha} R_k^{(n-3)(1-\alpha)} \leqslant \left(\sum_{k=1}^{N} R_k\right)^{\alpha} \left(\sum_{k=1}^{N} R_k^{n-3}\right)^{1-\alpha}$$

and

$$N = \sum_{k=1}^{N} 1 = \sum_{k=1}^{N} R_k^{1-\alpha} R_k^{(n-3)\alpha} \leqslant \left(\sum_{k=1}^{N} R_k\right)^{1-\alpha} \left(\sum_{k=1}^{N} R_k^{n-3}\right)^{\alpha}.$$

So

$$N \sum_{k=1}^{N} R_k^{n-2} \leqslant \left(\sum_{k=1}^{N} R_k\right) \left(\sum_{k=1}^{N} R_k^{n-3}\right)$$

and

$$R_* = \frac{\sum_{k=1}^{N} R_k^{n-2}}{\sum_{k=1}^{N} R_k^{n-3}} \leqslant \frac{\sum_{k=1}^{N} R_k}{N}. \tag{5.11}$$

The case for $n = 3$ is obvious.

If $n > 3$, then we need to prove

$$\left(\sum_{k=1}^{N} R_k\right) \left(\sum_{k=1}^{N} R_k^{n-3}\right) \leqslant N \left(\sum_{k=1}^{N} R_k^{n-2}\right). \tag{5.12}$$

It can be done by rewriting

$$\sum_{k=1}^{N} R_k = \sum_{k=1}^{N} 1^{\alpha} R_k^{(n-2)(1-\alpha)}, \quad \text{and}$$
$$\sum_{k=1}^{N} R_k^{n-3} = \sum_{k=1}^{N} 1^{1-\alpha} R_k^{(n-2)\alpha}$$

for $\alpha = (n-3)/(n-2)$.

(b.) These inequalities can be proved in a similar way.

(c.) Equation (5.9) can be rewritten as

$$\sum_{k=1}^{N} R_k^{n-1} \leqslant N^{1/n} \left(\sum_{k=1}^{N} R_k^n \right)^{1-1/n}. \tag{5.13}$$

This is the case when $f_k = 1, g_k = R_k^n$ and $\alpha = 1/n$ in (5.10). □

Combining equations (5.8), (5.9) and the energy estimate (5.4), we obtain the following further estimates.

Theorem 5.2. *There exist positive constants C_1 and C_2 depending only on the space dimension n such that for any solution $\{R_k\}$ of the mean field model for Ostwald ripening (4.1), we have*

$$\int_0^T R_*(t)^{-2}\, dt \geqslant C_1 \int_0^T t^{-2/3}\, dt \tag{5.14}$$
$$\int_0^T \left(\frac{\sum_{k=1}^{N} R_k}{N(t)} \right)^{-2} dt \geqslant C_1 \int_0^T t^{-2/3}\, dt \tag{5.15}$$
$$\int_0^T \left(\frac{\sum_{k=1}^{N} R_k^n}{N(t)} \right)^{-2/n} dt \geqslant C_1 \int_0^T t^{-2/3}\, dt \tag{5.16}$$

for all $T \geqslant C_2 S_{Ost}(0)^3$.

Remark. Equations (5.14), (5.15) and (5.16) indicate that the critical and average radii can not grow faster than the $t^{1/3}$ power law and the average mass can not grow faster than the $t^{n/3}$ power law, in the averaged sense. Furthermore, since the total mass $\sum_{k=1}^{N} R_k^n$ is conserved, equation (5.16) indicates that the particle number N can not decrease faster than the $t^{-n/3}$ power law. These results also show that critical and average radii are both possible choices of typical length scales that characterize the coarsening behavior. In the 3D case, they are equal and this partially justifies the prediction of the LSW theory about the coarsening rates in Ostwald ripening.

6 Discussions and Conclusions

We have studied a mean field model for the ripening of droplets in the late stage of a thin film when the underlying substrate is one dimensional. It is shown that this mean field model is equivalent to the classical LSW model for two dimensional phase transitions.

When the underlying substrate is two dimensional, the dynamics of the thin film is more complicated and we are currently studying the possibility of such a mean field approach.

On the other hand, in this mean field model, the collision of droplets are ignored. We expect to generalize the model to include collisions, which will possibly result in a mixture of LSW mean field model and Smoluchowshi coagulation model. Rigorous derivation and properties of such generalized models are to be explored.

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