Name: Solutions

Section:

I.1-I.2		II	
I.3-I.4		III	
		Total	

MA 2051 B 2013 – Test 3

Instructions: Do your work on this paper. Put your **name** and **section number** above. Work neatly. Show your work. **DO NOT SKIP STEPS. JUSTIFY YOUR ANSWERS.**[1] Brains only — no calculators, books, scrap paper, etc.

40 pts **I.** Let $g(t) = \begin{cases} 3t, & 0 \le t < 5 \\ 0, & 5 \le t < \infty \end{cases}$

1. Use the definition of Laplace transform to write $\mathcal{L}\{g(t)\}$ as an integral.

$$\boxed{\mathcal{L}\{g(t)\} = \int_0^\infty g(t)e^{-st}\,dt = 3\int_0^5 te^{-st}\,dt}$$

2. Evaluate that integral to find $G(s) = \mathcal{L}\{g(t)\}$.
Hint: $\int te^{kt}\,dt = \frac{t}{k}e^{kt} - \frac{1}{k}\int e^{kt}\,dt.$

Hint w/ $k=-s \Rightarrow 3\int_0^5 te^{-st}\,dt = \frac{3te^{-st}}{-s}\Big|_{t=0}^{5} + \frac{3}{s}\int_0^5 e^{-st}\,dt$

$$= \frac{-15e^{-5s}}{s} - \frac{3}{s\,s}e^{-st}\Big|_{t=0}^{5} = -\frac{15}{s}e^{-5s} - \frac{3}{s^2}(e^{-5s}-1)$$

$$= \boxed{\frac{3}{s^2}(1-e^{-5s}) - \frac{15}{s}e^{-5s}}$$

[1] "Justify your answers" requires providing enough justification to permit a Calc IV student to follow your work without having taken this course. You can assume that such a student could look up specialized terms and definitions as needed.

Name:______________________________

3. Write $g(t)$ in terms of the unit step function.

$$g(t) = \begin{cases} 3t, & 0 \le t < 5 \\ 0, & 5 \le t < \infty \end{cases}$$

"Bump"

$$1 - \mathcal{U}(t-5) = \begin{cases} 1, & 0 \le t < 5 \\ 0, & 5 \le t \end{cases}$$

$\mathcal{U}(t-5)$

$$\therefore g(t) = 3t[1 - \mathcal{U}(t-5)] + 0 \cdot \mathcal{U}(t-5)$$

$$\boxed{g(t) = 3t[1 - \mathcal{U}(t-5)]}$$

4. Use the table of Laplace transforms provided to find the transform of the expression you found in problem 3.

$$\mathcal{L}\{3t[1 - \mathcal{U}(t-5)]\} = 3\mathcal{L}\{t\} - 3\mathcal{L}\{\underbrace{t}_{= f(t-5)}\mathcal{U}(t-5)\} \quad a = 5$$

$$f(\underbrace{t-5}_{u}) = t \Rightarrow f(u) = u+5 \Rightarrow F(s) = \mathcal{L}\{u+5\} = \frac{1}{s^2} + \frac{5}{s}$$

$$\Rightarrow t = u + 5$$

$$\therefore \mathcal{L}\{g(t)\} = \frac{3}{s^2} - 3e^{-5s}\left[\frac{1}{s^2} + \frac{5}{s}\right]$$

$$= \boxed{\frac{3}{s^2}\left(1 - e^{-5s}\right) - \frac{15}{s}e^{-5s}}$$

Extra credit (1 pt): Check your work by comparing your answers to problems 2 and 4.

$\Rightarrow$ #4 = #2 - Check ✓

Name:______________________________

40 pts **II.** Use the table of Laplace transforms provided to find the inverse of each of the following transforms.

1. $\frac{s}{s^2-6s+13} = \frac{s}{(s-3)^2-9+13} = \frac{s}{(s-3)^2+4}$

$[\![b^2-4ac = 36-52<0 \Rightarrow$ complex roots $\Rightarrow$ comp. sqr. $]\!]$

$$= \frac{s-3}{(s-3)^2+4} + \frac{3}{2}\frac{\cancel{3}\,2}{(s-3)^2+4} = F(s-3) + \frac{3}{2}G(s-3)$$

$[\![$ ① $= F(s-3)$ w/ $F(s) = \frac{s}{s^2+4} \Rightarrow f(t) = \cos 2t$

② $= \frac{3}{2}G(s-3)$ w/ $G(s) = \frac{2}{(s-3)^2+4} \Rightarrow g(t) = \sin 2t$ $]\!]$

$$= \mathcal{L}\{e^{3t}\cos 2t\} + \frac{3}{2}\mathcal{L}\{e^{3t}\sin 2t\}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2-6s+13}\right\} = e^{3t}\cos 2t + \frac{3}{2}e^{3t}\sin 2t}$$

2. $\frac{s}{s^2-5s+4} = \frac{s}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4} = \frac{A(s-4)+B(s-1)}{(s-1)(s-4)}$

$[\![b^2-4ac = 25-16 = 9 > 0 \Rightarrow$ real roots $\Rightarrow$ PFs $]\!]$

$$= \frac{(A+B)s - 4A - B}{(s-1)(s-4)} \Rightarrow \begin{array}{l} s^1:\ A+B=1 \Rightarrow \frac{3}{4}B = 1 \Rightarrow B = \frac{4}{3} \\ s^0:\ -4A-B=0 \Rightarrow A = -\frac{1}{4}B \\ \Rightarrow A = -\frac{1}{3} \end{array}$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2-5s+4}\right\}} = \mathcal{L}^{-1}\left\{\frac{-1/3}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{4/3}{s-4}\right\}$$

$$\boxed{= -\frac{1}{3}e^{+t} + \frac{4}{3}e^{+4t}}$$

Name:________________________________

20 pts **III.** 1. Use transform tables to find $\mathcal{L}\{t \cancel{\cos \pi t}\}$. $\sin 3t$ Use $\mathcal{L}\{t f(t)\} = -F'(s)$

$w/\ f(t) = \sin 3t \Rightarrow F(s) = \frac{3}{s^2+9} \Rightarrow F'(s) = \frac{-3 \cdot 2s}{(s^2+9)^2}$

$\therefore \boxed{\mathcal{L}\{t \sin 3t\} = -F'(s) = \frac{6s}{(s^2+9)^2}}$

2. Find the Laplace transform $X(s)$ of the solution of the spring-mass initial-value problem $x'' + 9x = 2\sin 3t$, $x(0) = 1$, $x'(0) = 0$. Do **NOT** invert the transform $X(s)$ to find $x(t)$!

$\mathcal{L}\{DE\} \Rightarrow s^2 X - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=0} + 9X = s^2 X - s + 9X = 2\mathcal{L}\{\sin 3t\} = \frac{2 \cdot 3}{s^2+9}$

$\therefore s^2 X - s + 9X = (s^2+9)X - s = \frac{6}{s^2+9}$

$\therefore \boxed{X(s) = \frac{s}{s^2+9} + \frac{6}{(s^2+9)^2}}$

Extra credit (up to 5 pts): Use the terms in the transform $X(s)$ you just found to predict the types of behavior would you expect to see in the solution $x(t)$. Be as precise as you can about growth or decay rates, periods or frequencies, etc.

$(s^2 + \underbrace{9}_{a^2})^{-1} \Rightarrow \sin 3t$ &/or $\cos 3t \Rightarrow$ oscillations w/ freq 3 rad/s

$(s^2+9)^{-2} \Rightarrow F'(s)$ w/ $F(s) = \frac{x}{(s^2+9)}$

$\therefore t \sin 3t$ &/or $t \cos 3t$ [see #2]

$\Rightarrow$ growing oscillations with freq 3 rad/s

(prop'l to t $\Rightarrow$ RESONANCE!)