Grading Guide

Name: Solutions

Section:__________

I.1-I.2		II	
I.3-I.4		III	
		Total	

MA 2051 B 2013 – Test 3

Instructions: Do your work on this paper. Put your **name** and **section number** above. Work neatly. Show your work. **DO NOT SKIP STEPS. JUSTIFY YOUR ANSWERS.**[1] Brains only — no calculators, books, scrap paper, etc.

40 pts **I.** Let $g(t) = \begin{cases} 3t, & 0 \le t < 5 \\ 0, & 5 \le t < \infty \end{cases}$

(10 each: 1-4)

10 1. Use the definition of Laplace transform to write $\mathcal{L}\{g(t)\}$ as an integral.

$$\boxed{\mathcal{L}\{g(t)\} = \int_0^\infty g(t)e^{-st}\,dt = 3\int_0^5 te^{-st}\,dt}$$

+5 +5

10 2. Evaluate that integral to find $G(s) = \mathcal{L}\{g(t)\}$.
Hint: $\int te^{kt}\,dt = \frac{t}{k}e^{kt} - \frac{1}{k}\int e^{kt}\,dt$.

Hint w/ $k=-s \Rightarrow 3\int_0^5 te^{-st}\,dt = \left.\frac{3te^{-st}}{-s}\right|_{t=0}^{5} + \frac{3}{s}\int_0^5 e^{-st}\,dt$

$= -\frac{15e^{-5s}}{s} - \frac{3}{5s}\left.e^{-st}\right|_{t=0}^{5} = -\frac{15}{s}e^{-5s} - \frac{3}{s^2}(e^{-5s}-1)$

$= \boxed{\frac{3}{s^2}(1-e^{-5s}) - \frac{15}{s}e^{-5s}}$

+7-10 for accurate integration, as w/ HW; if 1 is wrong, then check integration for ans. given. <7 if fundamental error

[1] "Justify your answers" requires providing enough justification to permit a Calc IV student to follow your work without having taken this course. You can assume that such a student could look up specialized terms and definitions as needed.

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10 3. Write $g(t)$ in terms of the unit step function.

$$g(t) = \begin{cases} 3t, & 0 \le t < 5 \\ 0, & 5 \le t < \infty \end{cases}$$

"Bump" $1 - \mathcal{U}(t-5) = \begin{cases} 1, & 0 \le t < 5 \\ 0, & 5 \le t \end{cases}$

$\mathcal{U}(t-5)$

$$\therefore\ g(t) = 3t[1 - \mathcal{U}(t-5)] + 0 \cdot \mathcal{U}(t-5)$$

$$\boxed{g(t) = 3t[1 - \mathcal{U}(t-5)]}$$

10 4. Use the table of Laplace transforms provided to find the transform of the expression you found in problem 3.

+2 for correct ID of f

$$\mathcal{L}\{3t[1-\mathcal{U}(t-5)]\} = 3\mathcal{L}\{t\} - 3\mathcal{L}\{\underbrace{t}_{=f(t-5)}\mathcal{U}(t-5)\}$$

$a=5$ +1

$f(t-5) = t \Rightarrow f(u) = u+5 \Rightarrow F(s) = \mathcal{L}\{u+5\}$

$u \Rightarrow t = u+5 \qquad = \frac{1}{s^2} + \frac{5}{s}$

+3 total

+7 total

$$\therefore\ \mathcal{L}\{g(t)\} = \frac{3}{s^2} - 3e^{-5s}\left[\frac{1}{s^2} + \frac{5}{s}\right]$$

$$= \boxed{\frac{3}{s^2}\left(1 - \frac{e^{-5s}}{s^2}\right) - \frac{15}{s}e^{-5s}}$$

Extra credit (1 pt): Check your work by comparing your answers to problems 2 and 4.

$\Rightarrow$ #4 = #2 - Check ✓ +1

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40 pts **II.** Use the table of Laplace transforms provided to find the inverse of each of the following transforms.

20 1. $\dfrac{s}{s^2-6s+13} = \dfrac{s}{(s-3)^2-9+13} = \dfrac{s}{(s-3)^2+4}$

+14 for overall correct approach

+6 for details

$[\![b^2-4ac = 36-52<0 \Rightarrow$ complex roots $\Rightarrow$ comp. sqr. $]\!]$

$= \dfrac{s-3}{(s-3)^2+4} + \dfrac{3}{2}\dfrac{\cancel{3}\,2}{(s-3)^2+4} = F(s-3) + \frac{3}{2}G(s-3)$

$[\![$ ① $= F(s-3)$ w/ $F(s) = \dfrac{s}{s^2+4} \Rightarrow f(t) = \cos 2t$

② $= \frac{3}{2}G(s-3)$ w/ $G(s) = \dfrac{2 = a}{(s-3)^2+4 = a^2} \Rightarrow g(t) = \sin 2t$ $]\!]$

$= \mathcal{L}\{e^{3t}\cos 2t\} + \frac{3}{2}\mathcal{L}\{e^{3t}\sin 2t\}$

$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-6s+13}\right\} = e^{3t}\cos 2t + \frac{3}{2}e^{3t}\sin 2t}$

20 2. $\dfrac{s}{s^2-5s+4} = \dfrac{s}{(s-1)(s-4)} = \dfrac{A}{s-1} + \dfrac{B}{s-4} = \dfrac{A(s-4)+B(s-1)}{(s-1)(s-4)}$

Same

+14 $[\![b^2-4ac = 25-16 = 9 > 0 \Rightarrow$ real roots $\Rightarrow$ PFs $]\!]$

+6 $= \dfrac{(A+B)s - 4A - B}{(s-1)(s-4)} \Rightarrow$ s^1: $A+B=1 \Rightarrow \frac{3}{4}B = 1 \Rightarrow B = \frac{4}{3}$

s^0: $-4A-B=0 \Rightarrow A = -\frac{1}{4}B \Rightarrow A = -\frac{1}{3}$

$\therefore \mathcal{L}^{-1}\left\{\dfrac{s}{s^2-5s+4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-1/3}{s-1}\right\} + \mathcal{L}^{-1}\left\{\dfrac{4/3}{s-4}\right\}$

$= -\frac{1}{3}e^{+t} + \frac{4}{3}e^{+4t}$

Overall $-+7$

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20 pts **III.** 10 1. Use transform tables to find $\mathcal{L}\{t\,\cancel{\cos \pi t}\ \sin 3t\}$. Use $\mathcal{L}\{t f(t)\} = -F'(s)$

w/ $f(t) = \sin 3t \Rightarrow F(s) = \dfrac{3}{s^2+9} \Rightarrow F'(s) = \dfrac{-3\cdot 2s}{(s^2+9)^2}$

$\therefore \boxed{\mathcal{L}\{t \sin 3t\} = -F'(s) = \dfrac{6s}{(s^2+9)^2}}$ 4 details +3

10 2. Find the Laplace transform $X(s)$ of the solution of the spring-mass initial-value problem $x'' + 9x = 2\sin 3t$, $x(0) = 1$, $x'(0) = 0$. Do **NOT** invert the transform $X(s)$ to find $x(t)$!

Overall +7 $\mathcal{L}\{DE\} \Rightarrow s^2 X - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=0} + 9X = s^2 X - s + 9X = 2\mathcal{L}\{\sin 3t\} = \dfrac{2\cdot 3}{s^2+9}$

Details +3 $\therefore s^2 X - s + 9X = (s^2+9)X - s = \dfrac{6}{s^2+9}$

$\therefore \boxed{X(s) = \dfrac{s}{s^2+9} + \dfrac{6}{(s^2+9)^2}}$

Extra credit (up to 5 pts): Use the terms in the transform $X(s)$ you just found to predict the types of behavior would you expect to see in the solution $x(t)$. Be as precise as you can about growth or decay rates, periods or frequencies, etc.

$(s^2+9)^{-1} \Rightarrow \sin 3t$ &/or $\cos 3t \Rightarrow$ oscillations w/ freq 3 rad/s

$9 = a^2$ +1 +1

$(s^2+9)^{-2} \Rightarrow F'(s)$ w/ $F(s) = \dfrac{x}{(s^2+9)}$

$\therefore t\sin 3t$ &/or $t\cos 3t$ [see #2]

$-+\cancel{3}\,2$ $\Rightarrow$ growing oscillations with freq 3 rad/s

(prop'l to $t \Rightarrow$ RESONANCE!)

+1