Name: Solutions

Section:

I.1-I.2		II	
I.3-I.4		III	
		IV	
		Total	

MA 2051 B 2013 – Test 2

Instructions: Do your work on this paper. Put your **name** and **section number** above. Work neatly. Show your work. **JUSTIFY YOUR ANSWERS.**[1] Brains only — no calculators, books, scrap paper, etc.

12 pts each **I.** Find a general solution of the given differential equation; if initial conditions are given, find its solution as well. Use results of other problems whenever possible. **Bonus:** +1 for each solution checked. (Not shown here.)

6.3.4/5 1. $2y'' - 3y' + y = 0$, $y(0) = 2$, $y'(0) = 1$. L, CC, H $\Rightarrow$ CE

$$y = e^{rx} \Rightarrow 2r^2\cancel{e^{rx}} - 3r\cancel{e^{rx}} + \cancel{e^{rx}} = 0 \Rightarrow 2r^2 - 3r + 1 = 0$$

$$r = \frac{-(-3) \pm \sqrt{(-3)^2 - 8}}{2 \cdot 2} = \frac{3 \pm 1}{4} = 1, 1/2$$

$$\therefore \boxed{y_g = C_1 e^{x} + C_2 e^{x/2}} \text{ GS}$$

$$\text{ICs} \Rightarrow \begin{cases} y(0) = C_1 + C_2 = 2 \\ y'(0) = C_1 + \frac{1}{2}C_2 = 1 \end{cases} \Rightarrow C_1 = 0,\ C_2 = 2$$

$$\therefore \boxed{y(x) = 2e^{x/2}} \text{ solves IVP}$$

2. $2y'' - 3y' + y = 2x - 1$. L, CC, NH $\Rightarrow$ CE + UDC

H: Use homo. sol'n from #1

P: Guess $y_p = Ax + B$: $2y_p'' - 3y_p' + y_p = 0 - 3A + Ax + B$

$y_p' = A$ $\qquad = Ax + (B - 3A) = 2x - 1$

$y_p'' = 0$ $\qquad A = 2$, $B - 3A = -1$

$\Rightarrow B - 3 \cdot 2 = -1$

$B = -1 + 6 = 5$

$\therefore y_p = 2x + 5$

$$\#1 \Rightarrow \boxed{y_g = 2x + 5 + C_1 e^{x} + C_2 e^{x/2}}$$

[1] "Justify your answers" requires providing enough justification to permit a Calc IV student to follow your work without having taken this course. You can assume that such a student could look up specialized terms and definitions as needed.

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~6.4.4/24

3. The spring-mass model $mx'' + kx = 0$, $x(0) = x_i$, $x'(0) = 0$. L, CC, H

$x = e^{rt} \Rightarrow mr^2 \cancel{e^{rt}} + k\cancel{e^{rt}} = 0 \Rightarrow mr^2 + k = 0 \Rightarrow r = \pm i\sqrt{\frac{k}{m}}$

$\therefore \boxed{x_g = C_1 \sin\sqrt{k/m}\,t + C_2 \cos\sqrt{k/m}\,t}$ GS

ICs: $x(0) = C_1 \cdot 0 + C_2 \cdot 1 = x_i \Rightarrow C_2 = x_i$

$x'(0) = \sqrt{k/m}\left(C_1\cos(0) - C_2\sin(0)\right) = \sqrt{k/m}\,C_1 = 0$

$\Rightarrow C_1 = 0$

$\therefore \boxed{x(t) = x_i \cos\sqrt{k/m}\;t}$

~6.6.4/32

4. $4x'' + 36x = 3\cos 2t$. L, CC, NH $\Rightarrow$ CE + UDC

H: #3 with $m = 4$, $k = 36 \Rightarrow \sqrt{k/m} = 3$

$\Rightarrow x_h = C_1 \sin 3t + C_2 \cos 3t$

P: Guess $x_p = A\sin 2t + B\cos 2t$ Note: $\omega = 2$ so no problem w/ x_h

$4x_p'' + 36x_p = 4(-4A\sin 2t - 4B\cos 2t)$
$+ 36(A\sin 2t + B\cos 2t)$

$= A(36-16)\sin 2t + B(36-16)\cos 2t$

$= 0 \cdot \sin 2t + 3 \cos 2t$

$\therefore A = 0,\ 20B = 3 \Rightarrow B = 3/20$

$\therefore x_p = \frac{3}{20}\cos 2t$

$\boxed{x_g = \frac{3}{20}\cos 2t + C_1\sin 3t + C_2\cos 3t}$

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18 pts **II.** Use your solution to problem 3, part I, the spring-mass model, to:

1. Show that the period of motion of the mass is independent of the initial displacement of the mass.
2. Show that the maximum displacement of the mass *does* depend upon the initial displacement of the mass. Does the maximum displacement depend upon anything else?

1) I.3 $\Rightarrow x(t) = x_i \cos\sqrt{k/m}\, t$ w/ $x(0) = x_i$, $x'(0) = 0$

Period T: $\sqrt{k/m}\, T = 2\pi \Rightarrow \boxed{T = 2\pi/\sqrt{k/m} = 2\pi\sqrt{\frac{m}{k}}}$

$\boxed{\text{Independent of } x_i}$

2) $\boxed{\text{Max of } x(t) \text{ is } |x_i|}$ because $|\cos\sqrt{\ }\, t| \leq 1$

(or $x'(t) = 0 \Rightarrow \ldots$)

$\boxed{\text{But } |x_i| \text{ is independent of } k, m, \text{ etc.}}$

So max depends <u>only</u> on x_i in this case.

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17 pts **III.** A brick of mass 6 kg hangs from the end of the spring. When the mass is at rest, the spring is stretched by 7 cm. The spring is then stretched an additional 2 cm and released from rest.

1. Find an expression for the spring constant k; recall that $g = 9.8$ m/s^2. (You need not calculate the actual value of k. Just write an expression I could evaluate with a calculator.)
2. Suppose this spring-mass system is subject to damping; i.e., the governing equation has the form $mx'' + px' + kx = 0$, where p is the damping coefficient. What is the smallest value of p that just prevents the mass oscillating? (You need not calculate the actual value of p. Just write an expression I could evaluate with a calculator.)

1) $m = 6$ stretches $7\text{cm} = .07\text{ m} \Rightarrow |F_s| = mg = k(.07)$ N

$\Rightarrow k = \frac{6g}{.07} = \boxed{\frac{6 \cdot 9.8}{.07} \text{ N/m} = k}$

2) $mx'' + px' + kx = 0$, $m = 6$ kg, k as in #1

To prevent oscillations, choose p : charac. roots are real.

$x = e^{rt} \Rightarrow mr^2 \cancel{e^{rt}} + pr\cancel{e^{rt}} + k\cancel{e^{rt}} = 0$

$\Rightarrow mr^2 + pr + k = 0 \Rightarrow r = \dfrac{-p \pm \sqrt{p^2 - 4mk}}{2p}$

So r is real if $p^2 - 4mk \geq 0$ or $p^2 \geq 4mk$

Hence, min value of p to avoid oscillation is $p^2 = 4mk$

or $p = 2\sqrt{mk} = 2\left(6 \cdot \frac{6 \cdot 9.8}{.07}\right)^{1/2} = \boxed{12\sqrt{9.8/.07} \text{ N/m/s}}$

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17 pts **IV.** Recall the approximate pendulum equation $L\theta'' + g\theta = 0$; L is the length of the pendulum, $g = 9.8$ m/s^2. Find an expression for L that gives the pendulum a period of exactly 2 seconds. How should I change that length to increase the period to 4 seconds?

$L\theta'' + g\theta = 0$

1. Find GS. 2. Calculate period. 3. Solve for L.

1. $\theta = e^{rt} \Rightarrow Lr^2 \cancel{e^{rt}} + g\cancel{e^{rt}} = 0 \Rightarrow Lr^2 + g = 0 \Rightarrow r = \pm i\sqrt{\frac{g}{L}}$

$\therefore \theta_g = C_1 \sin\sqrt{g/L}\, t + C_2 \cos\sqrt{g/L}\, t$

2. Period T of either term satisfies $\sqrt{g/L}\, T = 2\pi$

$\Rightarrow T = 2\pi\sqrt{\frac{L}{g}}$ s.

3. $T = 2$ s $\Rightarrow 2 = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{1}{\pi^2} = \frac{L}{g}$

$\Rightarrow \boxed{L = g/\pi^2 \text{ m for period of 2 s}}$

For $T = 4$ s, solve as in 3. Or note T prop'l to $\sqrt{L}$

$\Rightarrow$ double T to 4 s by increasing L by 4

$\therefore \boxed{L = 4g/\pi^2 \text{ m for period of 4 s}}$