Name: Solutions

Section:

I.1-I.2		II	
I.3-I.4		III	
		IV	
		Total	

MA 2051 B 2013 – Test 1

Instructions: Do your work on this paper. Put your **name** and **section number** above. Work neatly. Show your work. **JUSTIFY YOUR ANSWERS.**[1] Brains only — no calculators, books, scrap paper, etc.

12 pts each **I.** If appropriate, find a general solution of the given differential equation or a solution of the given initial-value problem. If a general solution is not appropriate, find the best solution you can and explain. **Bonus:** +1 for each solution checked.

easy version of 4.4.5/21(a) - b or 1.5/13(d)

1. $\frac{dy}{dt} + ay = b$, a and b constants. L, CC - find gen'l sol'n via CE, UDC

Homo via CE: $y_h' + ay_h = 0$, $y_h = e^{rt}$, r = const.

$\Rightarrow re^{rt} + ae^{rt} = 0 \Rightarrow r = -a \Rightarrow y_h = e^{-at}$

Part via UDC: $f = bt^0 \Rightarrow y_p = A_0$

$y_p' + ay_p = b \Rightarrow A_0' + aA_0 = b \Rightarrow aA_0 = b \Rightarrow A_0 = b/a$

$\Rightarrow y_p = b/a$

Check: $(Ce^{-at} + b/a)' + a(Ce^{-at} + b/a) = \cancel{-aCe^{-at}} + \cancel{aCe^{-at}} + b = b$ ✓

$\therefore \boxed{y_g = Ce^{-at} + b/a}$

SolveOrder1/6

2. $\frac{dy}{dt} + t^2 y = 0$, $y(0) = 2$ L, not CC - Sep Var. because H

SofV: $\int \frac{dy}{y} = -\int t^2\,dt \Rightarrow \ln|y| = -t^3/3 + C_1 \Rightarrow |y| = \underbrace{e^{C_1}}_{C} e^{-t^3/3}$

$\therefore y = Ce^{-t^3/3}$

IC: $y(0) = 2 \Rightarrow 2 = Ce^0 \Rightarrow C = 2$ $\therefore \boxed{y(t) = 2e^{-t^3/3}}$

[1] "Justify your answers" requires providing enough justification to permit a Calc IV student to follow your work without having taken this course. You can assume that such a student could look up specialized terms and definitions as needed.

Check 2: IC: $y(0) = 2e^0 = 2$ ✓

DE: $(2e^{-t^3/3})' + t^2 2e^{-t^3/3}$

$= \cancel{-t^2 2e^{-t^3/3}} + \text{"} = 0$ ✓

Name: Solutions

Solve Order 1/3 − or 4.4.5/21(a)

3. $\frac{du}{dt} - u = 1 - t$ L, CC, RHS $= 1-t \Rightarrow u_g$ via CE, UDC

Homo: $u_h' - u_h = 0$ ⊕ #1 w/ $a = -1 \Rightarrow u_h = e^t$

OR: $u_h = e^{rt}$, r = const. $\Rightarrow r\cancel{e^{rt}} - \cancel{e^{rt}} = 0 \Rightarrow r = 1 \Rightarrow u_h = e^t$

Part: $f = 1 - t \Rightarrow u_p = A_0 + A_1 t$

$u_p' - u_p = 1 - t \Rightarrow A_1 - (A_0 + A_1 t) = 1 - t \Rightarrow (A_1 - A_0) - A_1 t = 1 - t$

$\therefore A_1 - A_0 = 1,\ -A_1 = -1$

$\therefore A_1 = +1,\ A_0 = A_1 - 1 = 0$ $\Rightarrow u_p = t$

$\therefore \boxed{u_g = Ce^t + t}$

Check: $(Ce^t + t)' - (Ce^t + t)$
$= \cancel{Ce^t} + 1 - \cancel{Ce^t} - t = 1 - t$ ✓

4.4.4/6

4. $\frac{du}{dt} - u = 2e^t$, $\cancel{x}\ u(0) = 4$ L, CC, RHS = exp'l $\Rightarrow u_g$ via CE, UDC

Homo: $u_h' - u_h = 0$ ⊕ #3 $\Rightarrow u_h = e^t$

Part: $f = 2e^t \Rightarrow u_p = Ae^t$ [Homo. sol'n in guess

$\Rightarrow$ Better guess $u_p = t(Ae^t)$

[OR: $u_p = Ae^t \Rightarrow u_p' - u_p = 0 \stackrel{?}{=} 2e^t$
$\Rightarrow$ Better guess]

$[t(Ae^t)]' - [t(Ae^t)] = Ae^t + \cancel{tAe^t} - \cancel{tAe^t}$

$= Ae^t = 2e^t \Rightarrow A = 2$

$\therefore u_p = 2te^t$

$u_g = Ce^t + 2te^t$

$u_g(0) = 4 \Rightarrow 4 = C + 0 \Rightarrow \boxed{u(t) = 4e^t + 2te^t}$ solves IVP

 Check: IC: $u(0) = 4e^0 + 2 \cdot 0 \cdot e^0 = 4 \cdot 1 = 4$ ✓

DE: $(4e^t + 2te^t)' - (4e^t + 2te^t)$
$= \cancel{4e^t} + 2e^t + \cancel{2te^t} - \cancel{4e^t} - \cancel{2te^t} = 2e^t$ ✓

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18 pts

compare 2.3/9

II. The population of a certain group of cells increases from 10,000 to 22,000 in 21 days. Suppose this population $P(t)$ satisfies a differential equation of the form $\frac{dP}{dt} = kP$.

1. Find an expression for the numerical value of the birth rate k. (You need *not* evaluate the expression.)
2. How many cells would you expect to reproduce during day 1?

1. ① Solve $P' = kP$, $P(0) = 10{,}000$. ② Use $P(21) = 22{,}000$ to find k

① $P = e^{rt} \Rightarrow re^{rt} = kP \Rightarrow r = k \Rightarrow P = Ce^{kt}$ (or use S of V)

$P(0) = 10{,}000 \Rightarrow P(t) = 10{,}000\, e^{kt}$

② $P(21) = 10{,}000\, e^{21k} = 22{,}000 \Rightarrow e^{21k} = 22/10 = 2.2$

$\therefore 21k = \ln 2.2 \Rightarrow \boxed{k = \frac{1}{21}\ln 2.2}$ day^{-1}

2. Increase amount $\approx$ rate $\times$ $\underbrace{\text{elapsed time}}_{=1 \text{ day}}$

$\hookrightarrow = kP(1) = k\,10{,}000$

$\therefore \sim \boxed{\frac{\ln 2.2}{21} 10{,}000 \text{ repro. on day 1}}$

Other approx. approaches to 1: estimate P' somehow, then estimate $k \cong P'/P$; e.g., $\frac{\Delta P}{\Delta t} = \frac{22{,}000 - 10{,}000}{21}$,

$P = 10{,}000 \Rightarrow k \approx \frac{22{,}000 - 10{,}000}{21 \cdot 10{,}000} = \frac{11}{210}$ day^{-1}

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17 pts **III.** When Felix Baumgartner fell from his capsule toward the earth, he was subject to the forces of gravity and of air resistance. A simple differential equation modeling his velocity $v(t)$ is $\frac{dv}{dt} + \frac{k}{m}v = -g$. (Notation: $g = 9.8\,\mathrm{m/sec}^2$, m mass, k constant for air resistance; positive velocities are upward.)

1. Find the steady-state (constant) solution v_{ter}, the so-called terminal velocity. Is v_{ter} positive or negative? Is its sign reasonable physically?
2. Use a solution formula to show that $\lim_{t\to\infty} v(t) = v_{\text{ter}}$, regardless of the value of $v(0)$. (**Hint:** you can use the result of problem I.1.) Is that mathematical result good news or bad news for Felix?

1. $v_{ter} = \text{const.} \Rightarrow \overset{0}{\cancel{v_{ter}'}} + \frac{k}{m} v_{ter} = -g \Rightarrow \boxed{v_{ter} = \frac{-gm}{k} < 0}$

$v_{ter} < 0$ is reasonable - falling to earth at steady speed

2. Use I.1 with $u = v$, $a = \frac{k}{m}$, $b = -g$; Check v_{ter}: $(-gm/k)' + \frac{k}{m}\left(\frac{-gm}{k}\right) = 0 - g = -g$ ✓

$\Rightarrow v_g = Ce^{-kt/m} - \frac{mg}{k} = Ce^{-kt/m} + v_{ter}$

Regardless of value of C (determined by $v(0)$), $\lim_{t\to\infty} v_g(t) = v_{ter}$ because $Ce^{-kt/m} \to 0$

Good news: won't fall faster & faster

Bad news: v_{ter} may still be too fast for safety

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17 pts **IV.** Recall that the temperature $T(t)$ of an unheated house can be modeled by $\dfrac{dT}{dt} = -\dfrac{Ak}{cm}(T - T_{\text{out}})$. House 1 has $+Ak/cm = 0.01 \text{ min}^{-1}$. House 2 has $+Ak/cm = 0.02 \text{ min}^{-1}$. Suppose that the outside temperature is $T_{\text{out}} = 5°$ C and that both houses start at $T(0) = 25°$ C.

Which house's temperature drops faster initially, House 1 or House 2? Which house would you prefer to be in?

House 1: $T_1' = -0.01(25-5) = -0.2$ °C/min

" 2: $T_2' = -0.02($ " $) = -0.4$ °C/min

So $\boxed{\text{House 2 temp. drops faster}}$; $T_2' < T_1'$ at $t=0$.

$\boxed{\text{Would prefer House 1 - would stay warmer longer}}$