Focus: Solve $\begin{cases} \cdot \text{ Linear} \\ \cdot \text{ Constant Coeff.} \end{cases}$ DEs

Form: $b_1 y' + b_0 y = f(x)$

$b_1, b_0 = \text{const.}$

Goal: General Sol'n $y_g = C y_h + y_p$

Homo sol'n: $b_1 y_h' + b_0 y_h = 0$

Particular sol'n: $b_1 y_p' + b_0 y_p = f(x)$

I. Find $y_h$ via Charac. Eqn:

Guess $y_h = e^{rx}$, $r = \text{const.} \Rightarrow r b_1 e^{rx} + b_0 e^{rx} = 0$

CE: $b_1 r + b_0 = 0 \Rightarrow r = -b_0/b_1$

II. Consider $f(x) = \begin{cases} a_n x^n + \cdots + a_1 x + a_0 & \text{(poly)} \\ a e^{qx} & \text{(exp'l)} \\ a \cos px + b \sin px & \text{(sin/cos)} \end{cases}$

Guess $y_p$ = same type of func.

Famous Examples $a_1(x)y' + a_0(x)y = f(x)$

**BBC** $\frac{dT}{dt} = -(\underset{\text{Ak/cm}}{.28})(T - \underset{T_{out}}{30}), \quad T(0) = 5$

$$T' + .28T = \underbrace{+.28 \cdot 30}_{f(t)}, \quad T(0) = 5$$

$b_1 = 1 \quad b_0 = .28$

Forcing term (RHS) $= +8.4 = f$

- Particular sol'n solves: $T_p' + .28T_p = 8.4$

  Captures "external forcing"

  Guess: $T_p = D \Rightarrow D' + .28D = 8.4$

  $$D = \frac{8.4}{.28} = 30$$

  $$\therefore \boxed{T_p = 30}$$

- Homo. sol'n solves: $T_h' + .28T_h = 0$

  Captures "decay to abs. zero"

  Guess: $T_h = e^{rt} \Rightarrow re^{rt} + .28e^{rt} = 0$

  $$T_h = e^{-.28t}$$

Gen'l Sol'n: $T_g = \underset{\underset{-25}{\shortparallel}}{C}e^{-.28t} + 30$

$T(0) = 5 \Rightarrow$

Famous II

Lin, CCDE

Ireland $P' = .015P - .209$, $P(0) = 1.2$

$P' - .015P = -.209$, $P(0) = 1.2$

Homo: $P_h' - .015P_h = 0 \Rightarrow P_h = e^{.015t}$

Repro. w/out effects of emigration

$P_h \to \infty$ (no pop'n loss)

Particular: $P_p' - .015P_p = -.209$.

Emigration/external forcing acts

to reduce P: $P_p' \cdots = \ominus .209$

Guess $P_p = A$ (const.) $\Rightarrow$

$A' - .015A = -.209 \Rightarrow A = \frac{.209}{.015}$

$$P_g = \underbrace{Ce^{.015t}}_{P_h} + \underbrace{\frac{.209}{.015}}_{P_p}$$

# Method of Undetermined Coeff's

p. 168

**TABLE 4.1 Particular solutions via undetermined coefficients for the constant-coefficient, first-order linear equation $b_1 y' + b_0 y = f(x)$.**

$$b_1 y' + b_0 y = f(x)$$

| Forcing Term $f(x)$ | Trial Particular Solution $y_p(x)$ |
|---|---|
| $a_n x^n + \cdots + a_1 x + a_0$ | $A_n x^n + \cdots + A_1 x + A_0$ |
| $ae^{qx}$ | $Ae^{qx}$ |
| $a \cos px + b \sin px$ | $A \cos px + B \sin px$ |

If the assumed form of the particular solution solves the corresponding homogeneous equation, multiply the assumed form by $x$.

Lowercase letters $a$, $a_i$, $b$ are constants given in the forcing function. The coefficients to be determined are denoted by uppercase letters $A$, $A_i$, $B$.

1. Educated guess
2. Sub 'n' chug
3. Equate coeff's

Find Gen'l Sol'n of

$y' + 2y = 4$

Homo: $y_h' + 2y_h = 0$

$y_h = e^{rt} \Rightarrow re^{rt} + 2e^{rt} = 0 \Rightarrow r = -2$

$\therefore y_h = e^{-2t}$

Part: $y_p' + 2y_p = 4$ ← Poly order $n = 0$

$y_p = A_0 \Rightarrow \underbrace{A_0'}_{=0} + 2A_0 = 4 \Rightarrow A_0 = 2$

$\therefore y_p = 2$

Gen'l sol'n: $\boxed{y_g = Ce^{-2t} + 2}$

$y' + 2y = e^{-x}$

Homo: $y_h' + 2y_h = 0$

p. 3 $\Rightarrow$ $y_h = e^{-2x}$

Part: $y_p' + 2y_p = e^{-x}$

$y_p = Ae^{-x}$

$y_p' + 2y_p = (Ae^{-x})' + 2(Ae^{-x})$

$= -Ae^{-x} + 2Ae^{-x} = e^{-x}$

$Ae^{-x} = 1e^{-x}$

$A = 1$

$\therefore$ $y_p = e^{-x}$

Gen'l: $\boxed{y_g = Ce^{-2x} + e^{-x}}$

$\underline{y' + 2y = 3\cos 2x}$

<u>Homo</u>: $p.3 \Rightarrow y_h = e^{-2x}$

<u>Part</u>: $y_p' + 2y_p = 3\cos 2x$

$$y_p = A\cos 2x + B\sin 2x$$

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p' + 2y_p = -2A\sin 2x + 2B\cos 2x + 2(B\sin 2x + A\cos 2x) = 3\cos 2x$$

$$\therefore \underbrace{(-2A + 2B)}_{=0}\sin 2x + \underbrace{(2B + 2A)}_{=3}\cos 2x = 3\cos 2x$$

$$-2A + 2B = 0 \Rightarrow A = B$$

$$2B + 2A = 3 \Rightarrow 4A = 3 \Rightarrow A = \frac{3}{4} = B$$

$$\therefore y_p = \frac{3}{4}(\cos 2x + \sin 2x)$$

$$\boxed{\underline{\text{Gen'l}}:\ y_g = Ce^{-2x} + \frac{3}{4}(\cos 2x + \sin 2x)}$$

$y' + 2y = 4 + 3\cos 2x$

Previous work $\Rightarrow$

$$\boxed{y_g = Ce^{-2x} + 2 + \frac{3}{4}(\cos 2x + \sin 2x)}$$

p. 3 p. 3 p. 5

$$y' + 2y = 2e^{-2x}$$

Homo: $y_h' + 2y_h = 0 \overset{p.3}{\Rightarrow} y_h = e^{-2x}$

Part: $y_p' + 2y_p = 2e^{-2x}$

BAD $y_p = Ae^{-2x}$

$$(Ae^{-2x})' + 2(Ae^{-2x}) = 2e^{-2x}$$

$$-2Ae^{-2x} + 2Ae^{-2x} = 2e^{-2x}$$

Guess $e^{-2x} \Leftarrow 0 \overset{?}{=} 2e^{-2x}$

solves Homo. DE

GOOD: $y_p = xAe^{-2x}$

$$(xAe^{-2x})' + 2(xAe^{-2x}) = +2e^{-2x}$$

$$\underbrace{A}e^{-2x} + \cancel{(-2)xAe^{-2x}} + \cancel{2xAe^{-2x}} = \underbrace{2}e^{-2x}$$

equate coeff's

$A = 2$

$\therefore y_p = 2xe^{-2x}$

$$\boxed{\text{Gen'l } \; y_g = Ce^{-2x} + 2xe^{-2x}}$$