For Test 3

- Know $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$
  - use defn to calculate $\mathcal{L}$
- Be fluent w/ table of transforms -
  - *recognize "shapes", "styles"
- Fair questions
  - $\mathcal{L}\{f\}$ - table or defn
  - $\mathcal{L}^{-1}\{F\}$ - "
  - Solve IVP
  - Given $F(s)$, what behavior expected in $f(t) = \mathcal{L}^{-1}\{F(s)\}$? (Exponential growth/decay; oscillations; on-off - give numerical values of rate, freq., ...)

$\frac{a,\ s}{s^2+a^2} \rightarrow \sin/\cos(at)$

$F(s-a),\ \frac{1}{s-a} \leftrightarrow$

e^{-as}

Typo - 10.6/11 Solution NOT -2

$$g(t) = -2[1-\mathcal{U}(t-4)] + 4[\mathcal{U}(t-4) - \mathcal{U}(t-8)]$$
$$= -2 + 6\,\mathcal{U}(t-4) - 4\,\mathcal{U}(t-8)$$

- δ func.
- PFs vs. Comp. Squ.
- Write out $\mathcal{U}$

10.6/11: $g(t) = -2t^2[1-\mathcal{U}(t-4)] + 4[\mathcal{U}(t-4) - \mathcal{U}(t-8)]$
$\qquad + 0\cdot[\mathcal{U}(t-8)]$

$$\mathcal{L}\{g\} = \mathcal{L}\{-2t^2[1-\mathcal{U}(t-4)] + \cdots\}$$
$$= \mathcal{L}\{-2t^2\} + 2\mathcal{L}\{t^2\mathcal{U}(t-4)\}$$
$$= -2\,\frac{2!}{s^3} + 2\{\underbrace{t^2}_{=f(t-4)}\mathcal{U}(t-4)\} \quad \hookrightarrow a=4$$

$\Longrightarrow$ [[Use $f(t-4) = t^2$ to find $f(u) = \cdots$

$$f(\underset{\sim}{t-4}) = t^2 \Rightarrow f(u) = (u+4)^2$$
$$u = t-4 \Rightarrow t = u+4$$
$$\mathcal{L}\{f(u)\} = \mathcal{L}\{u^2+8u+16\} = \underbrace{\frac{2!}{s^3} + 8\frac{1}{s^2} + \frac{16}{s}}_{F(s)}$$
]]

$$\mathcal{L}\{g\} = \frac{-4}{s^3} + 2F(s)e^{-4s}$$
$$= \frac{-4}{s^3} + 2\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)e^{-4s}$$

10.6/11) $y' + 2y = g(t)$, $y(0) = 5$

$$\mathcal{Y}(s) = \mathcal{L}\{y\} \Rightarrow s\mathcal{Y} - \overset{=5}{y(0)} + 2\overset{\mathcal{Y}}{\cancel{y}} = G(s)$$

$$(s+2)\mathcal{Y} = 5 + G(s)$$

$$\mathcal{Y} = \frac{5}{s+2} + \frac{1}{s+2}\left[-\frac{4}{s^3} + \left(\frac{\cancel{8}4}{s^3} + \frac{16}{s^2} + \frac{32}{s}\right)e^{-4s}\right]$$

$$y(t) = \mathcal{L}^{-1}\{\mathcal{Y}\} =$$

10.5/5) $T' + 6T = \pi\delta(t-4)$

$$T(0) = 17$$

$$s\underbrace{\mathcal{L}\{T\}}_{Q} - \overset{=17}{T(0)} + 6\mathcal{L}\{T\} \quad \pi e^{-4s}$$

$$Q(s+6) = 17 + \pi e^{-4s}$$

$$Q = \frac{17}{s+6} + \pi\frac{e^{-4s}}{s+6}$$

The forced model (5.13–5.14), page 210, of section 5.1 applies with the position $h(t)$ of the upper end of the spring given by $h(t) = H\delta(t-a) - h(0)$ for some H. The model is

$$mx'' + kx = kH\delta(t-a), \quad x(0) = x_i, \quad x'(0) = v_i. \ ■$$

To grasp the physical meaning of an equation such as $mx'' + \cdots = kH\delta(t-a)$, informally integrate both sides. Then the relation is $mx' + \cdots = kH$. The quantity on the left, mx', is the change in the momentum of the mass, and kH represents that change. Since mx'' is a force (mass times acceleration), $kH\delta(t-a)$ is an *impulsive force* of brief duration and sufficient magnitude to produce a change in momentum of kH.

Closing a switch at time $t = a$ in a series RLC circuit can cause a step function change in potential, $E(t) = V\mathcal{U}(t-a)$. The current model $Li'' + Ri' + i/C = E'(t)$ becomes

$$Li'' + Ri + \frac{1}{C}i = V\delta(t-a),$$

where the impulse $V\delta(t-a)$ is the consequence of a step increase V in potential at $t = a$. The time integral of $V\delta(t-a)$ is the magnitude V of the voltage change.

10.5.1 Exercises

EXERCISE GUIDE

To gain experience . . .	Try exercises
With transforms involving $\delta(t-a)$	1–3, 5–7, 9–13
Solving initial-value problems	2, 5–7, 6–12
Analyzing models involving impulses	4–6, 9–12
With $\delta(t-a)$ as the derivative of $\mathcal{U}(t-a)$	7–8, 11

1. Find $\mathcal{L}^{-1}\{1\}$.

2. Solve $y' + ky = 4\delta(t-6)$, $y(0) = 2$.

3. Invert the transform $\dfrac{s}{s-2}$.

4. Sketch a graph of the solution $P(t) = P_i e^{kt} - Re^{k(t-a)}\mathcal{U}(t-a)$ of the population model $P' = kP - R\delta(t-a)$, $P(0) = P_i$, considered in example 29. Explain the significance of the jumps in the graph.

5. Consider the heat-loss model $T' + (Ak/cm)T = b\delta(t-a)$, $T(0) = 0$. Solve this initial-value problem. Use your solution to explain the significance of the parameter b. Of what physical situation might this initial-value problem be a model?

6. Solve the initial-value problem

$$mx'' + kx = kH\delta(t-a), \quad x(0) = x_i, \quad x'(0) = v_i,$$

developed in example 30. It models a spring-mass system in which the upper end of the spring is jerked suddenly. Consider $x_i = v_i = 0$. Use your solution to explain the significance of the parameter H.

In exercises 1–2, use the definition of Laplace transform to find the transform of the given function. Sketch the graph of each function before you attempt to find its transform.

1. $g(t) = \begin{cases} mt, & 0 \le t \le b \\ 0, & b < t < \infty \end{cases}$

2. $g(t) = \begin{cases} 0, & 0 \le t \le 2 \\ t-2, & 2 \le t \le 4 \\ 6, & 4 < t < \infty \end{cases}$

In exercises 3–10, find the indicated transform; a and ω are constants. Confirm each result with DELAB.

3. $\mathcal{L}\{\cos 2t \sin 4t\}$
4. $\mathcal{L}\{\cos^2 3t\}$
5. $\mathcal{L}\{e^{-4t}\cos 2t \sin 4t\}$
6. $\mathcal{L}\{e^{-2t}\cos^2 3t\}$
7. $\mathcal{L}\{t(1-t)\}$
8. $\mathcal{L}\{e^{at}\cos \omega t\}$
9. $\mathcal{L}\{t \cos \omega t\}$
10. $\mathcal{L}\{\sin \omega t + \omega t \cos \omega t\}$

In exercises 11–12,

(i) Graph the forcing term and write it in terms of the unit step function $\mathcal{U}$.
(ii) Solve the initial-value problem, using Laplace transforms.
(iii) Give the interval(s) within which the highest derivative of the solution appearing in the equation is continuous.
(iv) Verify that your solution satisfies the differential equation within each interval of continuity.
(v) As appropriate, confirm each transform and solution result with DELAB.

11. $y' + 2y = g(t)$, $y(0) = 5$,

$$g(t) = \begin{cases} -2t & 0 \le t < 4 \\ 4, & 4 \le t < 8 \\ 0, & 8 \le t < \infty \end{cases}$$

Bump
$1 - \mathcal{U}(t-4)$
$\mathcal{U}(t-4) - \mathcal{U}(t-8)$
$\mathcal{U}(t-8)$

12. $y'' - 9y = g(t)$, $y(0) = 4$, $y'(0) = 0$,

$$g(t) = \begin{cases} 0, & 0 \le t < 2 \\ 2, & 2 \le t < \infty \end{cases}$$

13. Assuming that it exists, find the Laplace transform of the solution of $x' + x = g(t)$, $x(0) = -3$, where $g(t)$ is the isolated pulse and cosine wave of exercise 7, page 555,

$$g(t) = \begin{cases} 1, & 0 \le t \le 2\pi \\ \cos t, & 2\pi \le t \le 7\pi/2 \\ 0, & 7\pi/2 < t < \infty \end{cases}$$

Work directly from the differential equation.

14. Draw a graph like that of figure 10.10, page 547, illustrating the "turn-on" property of the unit step function product $f(t-a)\mathcal{U}(t-a)t - a$ for the function appearing in example 24, page 549,

$$g(t) = \begin{cases} 0, & 0 \le t \le 2 \\ \sin t, & 2 < t < \infty \end{cases}$$

In exercises 15–24, predict the general behavior of the function of t whose transform is given, then find the function and compare it with your prediction. Comment on differences. As appropriate, confirm each result with DELAB.

15. $\dfrac{4}{s^2+5s+6}$

16. $\dfrac{4e^{-2s}}{s^2+5s+6}$

17. $\dfrac{2s}{s^2-4}$

18. $\dfrac{2s}{s^2+4}$

19. $\dfrac{2se^{-5s}}{s^2+4}$

20. $\dfrac{s-1}{(s+1)(s^2-4)}$

21. $\dfrac{4}{s^2+2s+5}$

22. $\dfrac{3}{s^3+3s^2+2s}$

23. $\dfrac{2s+6}{s^2+6s+18}$

24. $\dfrac{3s}{s^2-3s+10}$

In exercises 25–28, the transform of the response of system is given. In each case,

(i) Determine whether the response contains oscillatory terms. If so, give their frequencies.
(ii) Determine whether the response includes exponential decay or growth. If so, give the factor in the exponential (or the corresponding time constant).
(iii) Determine if the response contains any terms that switch on or off. If so, give the switching time(s).

Do *not* invert the transforms. (As appropriate, use DELAB to check your results.)

25. $\dfrac{s^2+1}{s^4-16}$

26. $\dfrac{s^2e^{-2s}}{(s^2+4)(s^2+4s}$

27. $\dfrac{2-4e^{-3s}}{(s-1)^3}$

28. $\dfrac{3}{(s-3)(s^2+12}$

Laplace transforms coefficient different troduce variables f $Y(s) = \mathcal{L}\{y(t)\}$, solve for $Y(s)$, $Z(s$ exercises 29–36. Co

29. $y' = y + z$, y
$z' = -y + z$, z

30. $y' = z$, $y(0) =$
$z' = y$, $z(0) =$

31. $y' = -4y + 2z$
$z' = -3y + 3z$

32. $y' = -4z$, $y(0$
$z' = -4y$, $z(0$

33. $y' = y/2 + \sqrt{}$
$z' = \sqrt{3}\,y/2 -$

34. $y' = z + 4\cos$
$z' = y$, $z(0) =$

35. $y' = y + 3z +$
$z' = 3y + z -$

36. $y' = y - z - 2$
$z' = 2y + 4z -$

10.7 ■ CH