MA 2051 - #23 Tu/10 Dec 13

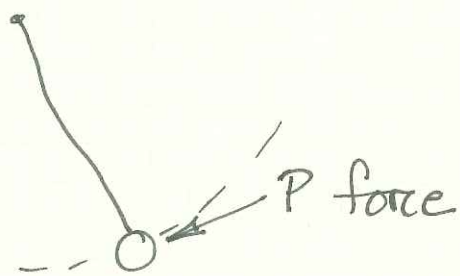
Unit Impulse Func:

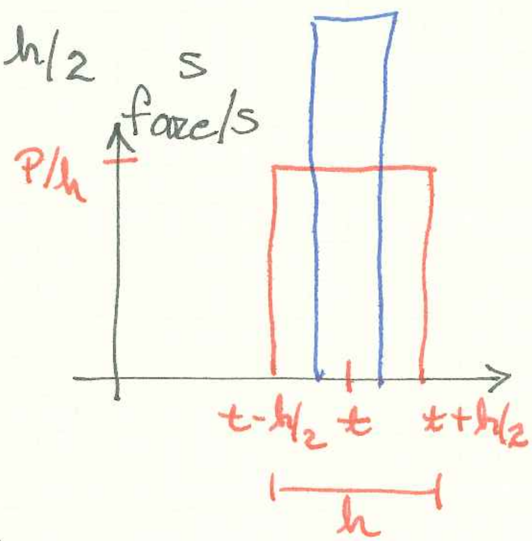
Finite action over infinitesimal time

Motivation:

Apply force of P/h N/s from

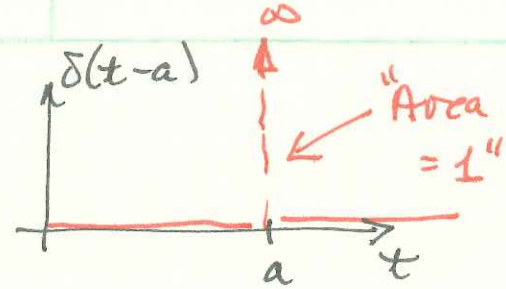
$t - h/2$ to $t + h/2$ s





Total force $= P$

If $h \to 0$, spike of force at time t



Delta "function" $\delta(t)$

Spike function

Properties

1. $\int_{-\infty}^{\infty} \varphi(t)\delta(t-a)\,dt = \varphi(a)$

2. $\delta(t-a) = 0, \quad t \neq a$

(Bizzare!)

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)e^{-st}\,dt = e^{-sa}$$

$$\mathcal{L}\{\delta(t)\} = 1$$

EXERCISE GUIDE

To gain experience . . .	Try exercises
Solving initial-value problems	8–11
Finding transforms of solutions of initial-value problems	12–13
Using the convolution theorem	14
Deriving the shifted function transform from the definition of Laplace transform	16

exercises 1–3, use $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$ with appropriate choice of f to find the Laplace transform of the ven function.

1. $e^{4t}\,\mathcal{U}(t-3)$

2. $(2-3t)\,\mathcal{U}(t-2)$

3. The isolated pulse $g(t) = \begin{cases} 0, & 0 \le t \le a \\ 1, & a < t \le b \\ 0, & b < t \end{cases}$

exercises 4–7,

(i) Rewrite the given function in terms of the unit step function $\mathcal{U}$.

(ii) Use that expression and $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$ to find the Laplace transform of the given function.

4. $g(t) = \begin{cases} 4t, & 0 \le t \le 7 \\ 0, & 7 < t < \infty \end{cases}$

5. $g(t) = \begin{cases} mt, & 0 \le t \le b \\ 0, & b < t < \infty \end{cases}$

6. $g(t) = \begin{cases} 0, & 0 \le t \le 2 \\ t-2, & 2 \le t \le 4 \\ 6, & 4 < t < \infty \end{cases}$

7. $g(t) = \begin{cases} 1, & 0 \le t \le 2\pi \\ \cos t, & 2\pi \le t \le 7\pi/2 \\ 0, & 7\pi/2 < t < \infty \end{cases}$

exercises 8–11,

(i) Graph the forcing term and write it in terms of the unit step function $\mathcal{U}$.

(ii) Solve the initial-value problem using Laplace transforms.

(iii) Give the interval(s) within which the highest derivative of the solution appearing in the equation is continuous.

(iv) Verify that your solution satisfies the differential equation within each interval of continuity.

8. $y' - 6y = g(t)$, $y(0) = 4$,

$$g(t) = \begin{cases} 0, & 0 \le t < 2 \\ 2, & 2 \le t < \infty \end{cases}$$

9. $y' + 2y = g(t)$, $y(0) = 0$,

$$g(t) = \begin{cases} 2t-1, & 0 \le t < 2 \\ 3, & 2 \le t < \infty \end{cases}$$

10. $y'' + 4y = g(t)$, $y(0) = 0$, $y'(0) = -2$,

$$g(t) = \begin{cases} -2, & 0 \le t < 4 \\ 4, & 4 \le t < 8 \\ 0, & 8 \le t < \infty \end{cases}$$

11. $P' - 0.015P = -E(t)$, $P(0) = 8$,

$$E(t) = \begin{cases} 0.209, & 0 \le t \le 4 \\ 0, & 4 < t < \infty \end{cases}$$

(This is the Irish potato-famine emigration model with population P in millions and time t in years measured from 1847. In this form, emigration stops in 1851. See equation (2.10), page 46, of section 2.2.)

In exercises 12–13, assume that the Laplace transform of the solution of each initial-value problem exists. Find the transform of the solution of the initial-value problem working directly from the differential equation.

12. $2y' - 5y = g(t)$, $y(0) = 3$, where $g(t)$ is the isolated pulse of exercise 3,

$$g(t) = \begin{cases} 0, & 0 \le t \le a \\ 1, & a < t \le b \\ 0, & b < t \end{cases}$$

13. $y' - ky = r(t)$, $y(0) = y_i$, where k is a constant and $r(t)$ is the ramp function of example 25,

$$r(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 < t < \infty \end{cases}$$

Ex: "Whapped" pendulum: $L\theta'' + g\theta = (P/m)\delta(t-3)$
$\theta(0) = \theta'(0) = 0$

Find $T(s) = \mathcal{L}\{\theta(t)\}$.

$$\mathcal{L}\{L\theta'' + g\theta\} = \mathcal{L}\{\tfrac{P}{m}\delta(t-3)\}$$

$$L\big(s^2T(s) - s\overset{=0}{\theta(0)} - \overset{=0}{\theta'(0)}\big) + gT(s)$$

$$= \frac{P}{m}e^{-3s}$$

$$\Rightarrow T(s)\,(Ls^2+g) = \frac{P}{m}e^{-3s}$$

$$T(s) = \frac{P}{m}\,\frac{1}{Ls^2+g}\,e^{-3s}$$

Ex: Solve "Whapped" pendulum IVP: $L\theta'' + g\theta = \frac{P}{m}\delta(t-3)$

$\theta(0) = \theta'(0) = 0$

Use $T(s) = \mathcal{L}\{\theta(t)\} = \frac{(P/m)}{Ls^2+g} e^{-3s}$

$$= \frac{(P/mL)}{\sqrt{g/L}} \underbrace{\frac{\sqrt{g/L}}{s^2 + g/L}}_{F(s)} e^{-3s}$$

Expectations:

1. $\theta(t) = 0$, $0 \le t < 3$ (Physics)
2. $\frac{\sqrt{\ }}{s^2+(\sqrt{\ })^2} \to \sin\sqrt{g/L}\, t$ (OK phys'ly)
3. $e^{-3s} \to t-3$, $u(t-3) = \begin{cases} 0 & t<3 \\ 1 & 3<t \end{cases}$

Time of whap ⟶ OK

Invert: Use: $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $a = \sqrt{g/L}$ (*)

(**) $\mathcal{L}\{f(t-b)u(t-b)\} = F(s)e^{-bs}$, $b=3$

$F(s) = \mathcal{L}\{\sin\sqrt{g/L}\, t\}$ (see (*))

In (**), $f(t) = \sin\sqrt{g/L}\, t$

$\Rightarrow f(t-b)u(t-b) = \sin\sqrt{\frac{g}{L}}(t-3)u(t-3)$

$$\therefore \boxed{\theta(t) = \left(\frac{P/mL}{\sqrt{g/L}}\right)\sin\sqrt{\frac{g}{L}}(t-3)\, u(t-3)}$$

The forced model (5.13–5.14), page 210, of section 5.1 applies with the position $h(t)$ of the upper end of the spring given by $h(t) = H\delta(t-a) - h(0)$ for some H. The model is

$$mx'' + kx = kH\delta(t-a), \quad x(0) = x_i, \quad x'(0) = v_i. \blacksquare$$

To grasp the physical meaning of an equation such as $mx'' + \cdots = kH\delta(t-a)$, informally integrate both sides. Then the relation is $mx' + \cdots = kH$. The quantity on the left, mx', is the change in the momentum of the mass, and kH represents that change. Since mx'' is a force (mass times acceleration), $kH\delta(t-a)$ is an *impulsive force* of brief duration and sufficient magnitude to produce a change in momentum of kH.

Closing a switch at time $t = a$ in a series RLC circuit can cause a step function change in potential, $E(t) = V\mathcal{U}(t-a)$. The current model $Li'' + Ri' + i/C = E'(t)$ becomes

$$Li'' + Ri + \frac{1}{C}i = V\delta(t-a),$$

where the impulse $V\delta(t-a)$ is the consequence of a step increase V in potential at $t = a$. The time integral of $V\delta(t-a)$ is the magnitude V of the voltage change.

10.5.1 Exercises

EXERCISE GUIDE

To gain experience ...	Try exercises
With transforms involving $\delta(t-a)$	1–3, 5–7, 9–13
Solving initial-value problems	2, 5–7, 6–12
Analyzing models involving impulses	4–6, 9–12
With $\delta(t-a)$ as the derivative of $\mathcal{U}(t-a)$	7–8, 11

1. Find $\mathcal{L}^{-1}\{1\}$.

2. Solve $y' + ky = 4\delta(t-6)$, $y(0) = 2$.

3. Invert the transform $\dfrac{s}{s-2}$.

4. Sketch a graph of the solution $P(t) = P_i e^{kt} - Re^{k(t-a)}\mathcal{U}(t-a)$ of the population model $P' = kP - R\delta(t-a)$, $P(0) = P_i$, considered in example 29. Explain the significance of the jumps in the graph.

5. Consider the heat-loss model $T' + (Ak/cm)T = b\delta(t-a)$, $T(0) = 0$. Solve this initial-value problem. Use your solution to explain the significance of the parameter b. Of what physical situation might this initial-value problem be a model?

6. Solve the initial-value problem

$$mx'' + kx = kH\delta(t-a), \quad x(0) = x_i, \quad x'(0) = v_i,$$

developed in example 30. It models a spring-mass system in which the upper end of the spring is jerked suddenly. Consider $x_i = v_i = 0$. Use your solution to explain the significance of the parameter H.