Your mid-term comments

⇒ 5% too little for paper HW
  ⊕ 10% for quizzes (≡ HW!) } 15%
  ⊕ Hour exams (≈ WW, HW)

⇒ More examples start to finish
  - Text, conf's for detailed examples
  - Lecture: balance strategy
    & $\underbrace{\text{mechanics}}_{\text{Will try for more.}}$

(See detail added after class →)

Details in text examples

At the end of the last section, we found the transform of the solution of the initial-value problem $y' + ky = e^{-3t}$, $y(0) = 4$. We begin with an example which inverts that transform.

■ EXAMPLE 6 *Find the inverse transform*

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)} + \frac{4}{s+k}\right\}.$$

Don't like parameter "k". Replace with a # - say 6.

Since $\mathcal{L}$ is linear, $\mathcal{L}^{-1}$ is linear as well. The inverse transform can be computed term by term, and constant factors can be brought outside the operator $\mathcal{L}^{-1}$, much like an antiderivative,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+k}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\}. \end{aligned}$$

To invert the second expression, search table 10.1 for a transform that looks like $1/(s+k)$. Entry 3 with $-a = k$ is the obvious choice. Since

$$\mathcal{L}\left\{e^{-kt}\right\} = \frac{1}{s-(-k)} = \frac{1}{s+k}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kt}.$$

The first term in the inverse is more of a problem, for we find nothing in the right-hand column of table 10.1 of the form

$$\frac{1}{(s+k)(s+3)}.$$

But we could invert the factors $1/(s+k)$ and $1/(s+3)$ if they stood alone. (We just found e^{kt} for the inverse of the first, and the second is the same with $k = 3$.)

Major steps

An expression such as $1/(s+k)(s+3)$ can be separated into its component fractions using *partial fractions*. (See appendix A.7.) For some unknown constants A and B, write

Partial-fraction decomposition

$$\begin{aligned} \frac{1}{(s+k)(s+3)} &= \frac{A}{s+k} + \frac{B}{s+3} \\ &= \frac{A(s+3) + B(s+k)}{(s+k)(s+3)}. \end{aligned}$$

Tactics

Since the first and last numerator must be identical, we have

$$1 = A(s+3) + B(s+k).$$

Equate powers of s.

Equations for A and B result from equating coefficients of like powers of s:

$$s^0: \quad 3A + kB = 1,$$
$$s^1: \quad A + B = 0.$$

Solve for partial fraction numerators.

Solving this pair of simultaneous equations yields

$$A = \frac{1}{3-k}, \qquad B = \frac{1}{k-3}.$$

Using these values of A and B, the inverse transform is

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)}\right\} &= A\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\ &= Ae^{-kt} + Be^{-3t} \\ &= \frac{e^{-kt} - e^{-3t}}{3-k}. \end{aligned} \tag{10.3}$$

Inverse transform, the solution of the initial-value problem

Combining the inverse transforms of the two individual terms, we obtain the desired inverse transform. The solution of $y' + ky = e^{-3t}$, $y(0) = 4$, is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} \\ &= \frac{e^{-kt}}{3-k} + \frac{e^{-3t}}{k-3} + 4e^{-kt} \\ &= \frac{(4k-13)e^{-kt} + e^{-3t}}{k-3}, \qquad k \neq 3. \ \blacksquare \end{aligned}$$

As this example illustrates, we can use Laplace transforms and their inverses to solve linear differential equations with constant coefficients. The steps are as follows:

LAPLACE TRANSFORM SOLUTION PROCESS

1. Transform the entire differential equation, using any initial values that are given. Solve for the transform of the solution.
2. Invert the transform to find the solution of the differential equation or initial-value problem.

Unfortunately, step 2 is easier said than done!

■ **EXAMPLE 7** *Illustrate these steps by solving the initial-value problem*

$$y' + ky = e^{-3t}, \quad y(0) = 4.$$

Transform differe
and solve for $Y(s$

Invert $Y(s)$ to fin
the initial-value p

Transform differe

Solve for $X(s)$.

Invert $X(s)$.

Comments from class discussion

- WW time consuming, frustrating, less relevant
  - ⇒ PD: use email more for help
  - ⇒ Will make due at midnight
- Post HW sol'ns
  - ⇒ will explore
- Quiz before HW returned
  - ⇒ Duu-h - argument for posting sol'ns

Please email or tell me other points
I missed.

Successful $\mathcal{L}^{-1}$

- Choose an approach
- Anticipate outcome: t func's

Ex: $\mathcal{L}^{-1}\left\{\dfrac{4}{s^2+2s-8}\right\}$: $s^2+2s-8=(s+4)(s-2)$

$$\frac{4}{s^2+2s-8}=\frac{4}{(s+4)(s-2)}=\begin{cases} ① \; 4\left(\frac{1}{s+4}\right)\left(\frac{1}{s-2}\right) \\ ② \; \frac{A}{s+4}+\frac{B}{s-2} \end{cases}$$

Approach ①: $F(s)=4\left(\frac{1}{s+4}\right) \Rightarrow f(t)=4e^{-4t}$

Conv. Thm

$$G(s)=\frac{1}{s-2} \Rightarrow g(t)=e^{2t}$$

Convolution Thm: $\mathcal{L}^{-1}\left\{\frac{4}{s^2+2s-8}\right\}=\int_0^t f(r)g(t-r)\,dr$

$$=\int_0^t 4e^{-4r}e^{2(t-r)}\,dr = 4e^{2t}\int_0^t e^{\underbrace{(-4r-2r)}_{-6r}}\,dr$$

$$=4e^{2t}\left(\frac{-1}{6}\right)e^{-6r}\Big|_{r=0}^{t}=-\frac{2}{3}e^{2t}\left(e^{-6t}-1\right)$$

$$=\boxed{-\frac{2}{3}\left(e^{-4t}-e^{2t}\right)}$$

Approach ②: $\dfrac{A}{s+4}+\dfrac{B}{s-2}=\dfrac{A(s-2)+B(s+4)}{(s+4)(s-2)}=\dfrac{(A+B)s-2A+4B}{(s+4)(s-2)}$

Equate powers of s in numerators $\begin{cases} s^1: A+B=0 \\ s^0: -2A+4B=4 \end{cases}$

$\Rightarrow A=-B;\ -6A=4 \Rightarrow A=-\frac{2}{3}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{4}{s^2+2s-8}\right\}=-\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}-\frac{1}{s-2}\right\}=\boxed{-\frac{2}{3}\left(e^{-4t}-e^{2t}\right)}$$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{4s}{s^2+2s-8}\right\}$

$s^2+2s-8=(s+4)(s-2)$

Same denominator so same approaches:

Con. Thm. OR Partial Frac's

Use Partial Frac's: $\dfrac{4s}{s^2+2s-8}=\dfrac{A}{s+4}+\dfrac{B}{s-2}$

$$\Rightarrow 4s = \underbrace{(A+B)}_{=4}s \underbrace{-2A+4B}_{=0} \Rightarrow A=2B$$

$$\Rightarrow 3B=4 \Rightarrow B=4/3,\ A=8/3$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{4s}{s^2+2s-8}\right\} = \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}+\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \boxed{\frac{4}{3}\left(2e^{-4t}+e^{2t}\right)}$$

Use Convolution Thm: $\dfrac{4s}{s^2+2s-8} = \overbrace{4\left(\dfrac{s}{s+4}\right)}^{F(s)}\overbrace{\left(\dfrac{1}{s-2}\right)}^{G(s)}$

$$G(s)=\frac{1}{s-2} \Rightarrow g(t)=e^{2t}$$

$$F(s)=4\frac{s}{s+4} = s\underbrace{\left(\frac{4}{s+4}\right)}_{H} = sH(s)$$

$$H \Rightarrow h(t)=4e^{-4t}$$

Use $sH(s) = \mathcal{L}\{h'(t)\}+h(0)$

$$= \mathcal{L}\{-16e^{-4t}\}+1$$

$$= \quad '' \quad + \underbrace{\mathcal{L}\{\delta(t)\}}$$

Whoops - haven't got to δ yet!

So stick w/ Partial Frac's!

Ex: $\mathcal{L}^{-1}\left\{\frac{4}{s^2+2s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+7}\right\}$

$b^2-4ac = 4-32<0 \Rightarrow$ complex roots $\Rightarrow$ Complete Square

Idea of Complete Square: $(s+b)^2 = s^2+2sb+b^2$

$$\Rightarrow s^2 + \underbrace{2s}_{2b} + 8 = \left(s+\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 8$$

$$= (s+1)^2 + 7$$

① Identify $F(s), f(t)$ ② Use $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

① $\frac{4}{(\ \ )^2+7}$ "looks like" $F(s) = \frac{4}{s^2+(\sqrt{7})^2}$

$$F(s) = 4\cdot\frac{1}{\sqrt{7}}\cdot\frac{\overbrace{\sqrt{7}}^{a}}{s^2+\underbrace{(\sqrt{7})^2}_{a^2}} = \frac{4}{\sqrt{7}}\mathcal{L}\{\sin\sqrt{7}t\}$$

$$\therefore f(t) = \frac{4}{\sqrt{7}}\sin\sqrt{7}t$$

② $F(s) = \frac{4}{s^2+(\sqrt{7})^2} \Rightarrow F(s\underbrace{+1}_{a=-1}) = \frac{4}{(s+1)^2+7}$

Shifted Trans. w/ $a=-1 \Rightarrow\ = \mathcal{L}\{e^{-t}f(t)\}$

$$= \mathcal{L}\left\{e^{-t}\frac{4}{\sqrt{7}}\sin\sqrt{7}t\right\}$$

$$\therefore \boxed{\mathcal{L}^{-1}\left\{\frac{4}{s^2+2s+8}\right\} = e^{-t}\frac{4}{\sqrt{7}}\sin\sqrt{7}t}$$