<u>Recall</u>: math. models to understand $v(t)$ in free fall

1. No air resistance

$$v \stackrel{?}{=} -gt + C \qquad \xrightarrow{\text{Check}} \qquad \frac{dv}{dt} = -g, \quad v(0) = v_i$$

2. Include air resistance prop'l to $-v$

$$v = \qquad \frac{dv}{dt} = -g - \left(\frac{k}{m}\right)v, \quad v(0) = v_i$$

<u>Terms</u>:

Ordinary Diff'l Eqn.

Initial-value Problem

Initial Condition

De~~In~~dependent Variable (unknown)

In~~De~~pendent Variable t

Sol'n of DE
" " IVP } Sol'n = function $v(t)$
"formula"

Check: plug-in

<u>Example</u>: Find formula for sol'n of IVP

$$\frac{dv}{dt} = v' = -g - (k/m)v, \quad v(0) = v_i$$

1. Solve DE by <u>Separation of Var's</u>

$$\int \frac{dv}{-g-(k/m)v} = \int dt = t + C$$

$\vdots$

(Compare Ex. 11, p.16)

$\vdots$

$$v(t) = \frac{-gm}{k} + Ce^{-(k/m)t}$$

2. Use IC to find C

$$v(0) = \frac{-gm}{k} + C = v_i$$

$$\Rightarrow C = v_i + \frac{gm}{k}$$

$\therefore$ Sol'n of IVP $v(t) = \frac{-gm}{k} + \left(\frac{gm}{k} + v_i\right)e^{-\frac{kt}{m}}$

<u>Note</u>: $v \to \frac{-gm}{k}$ as $t \to \infty$

3. Check sol'n of IVP

① $v(0) = v_i$ ✓

② $\frac{d}{dt}\left[-\frac{gm}{k} + \left(\frac{gm}{k} + v_i\right)e^{-kt/m}\right] = -g - \frac{k}{m}[\ldots]$

Population Model:

Law: conservation - no magical add/remove

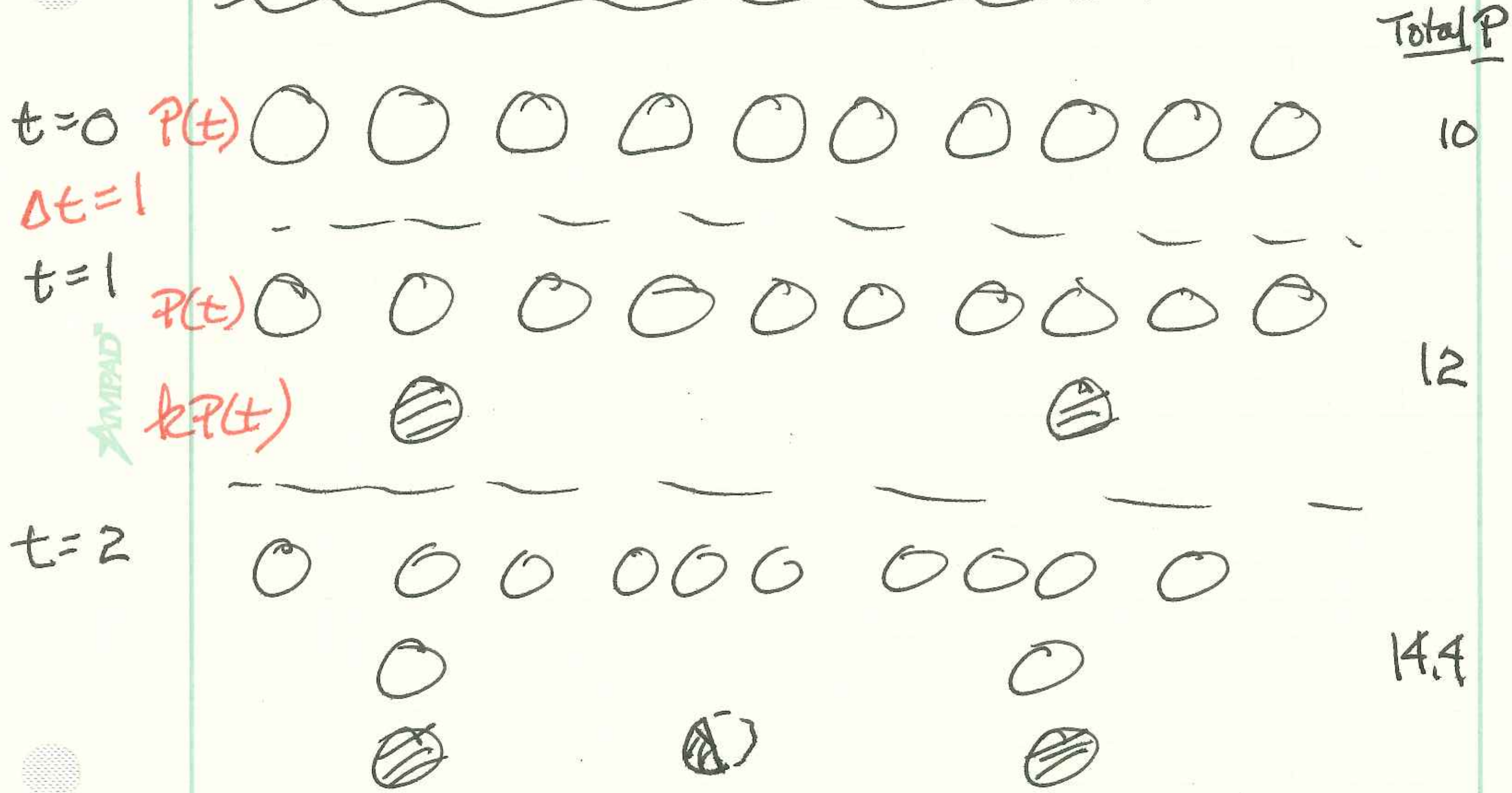
"Fact": fixed fraction (say, .02) reproduces per unit time

DE for pop'n $P(t)$

Rate of change of $P = \overset{.2}{\cancel{k}}\,P$

$$\boxed{\frac{dP}{dt} = .2P, \quad P(0) = P_i}$$

Fixed fraction reproduces/unit time : 20% = .2



More formal derivation:

$$P(t+\Delta t) - P(t) = \text{change in } P \quad t \to t+\Delta t$$

$$= \left(P(t) + kP(t)\Delta t\right) - \left(P(t)\right)$$

$$= kP(t)\Delta t$$

[rate × time = amount
kP Δt = "]

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = kP(t)\cancel{\Delta t}$$

$$\lim_{\Delta t \to 0}\left[\qquad\right]$$

$$\boxed{\frac{dP}{dt} = kP}$$

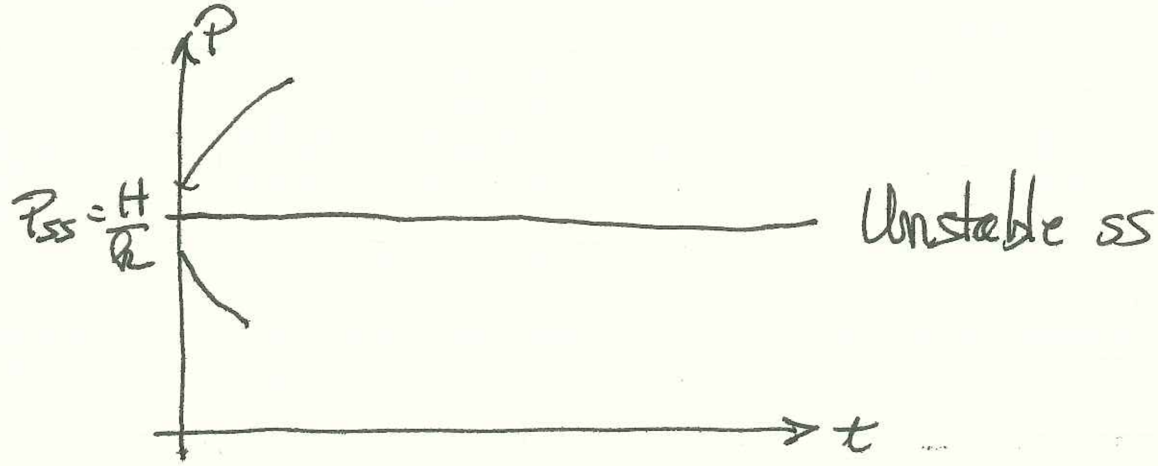
New situation:

- Const. fraction k reproduces per unit time
- Individuals removed (harvested) at rate H

$$\frac{P(t+\Delta t)-P(t)}{\Delta t} = \underbrace{kP(t)\cancel{\Delta t}}_{\text{Repro}} - \underbrace{H\cancel{\Delta t}}_{\text{Harvest}}$$

$$\lim_{\Delta t\to 0}[\,] \Rightarrow \boxed{\frac{dP}{dt} = kP - H,\ P(0)=P_0}$$

Analysis: $P' = kP - H \;,\; P(0) = P_i$



Easy sol'n: $P_{ss} = \text{const} \Rightarrow \overset{=0}{P_{ss}'} \overset{?}{=} kP_{ss} - H$

$$0 = kP_{ss} - H \Rightarrow P_{ss} = H/k$$

$$0 < P_i < H/k \Rightarrow P'(0) = kP_i - H = k\underbrace{\left(P_i - \frac{H}{k}\right)}_{\ominus} < 0$$

$$H/k < P_i \Rightarrow P'(0) > 0$$

Sol'n formula by Separation of Variables

$$\frac{dP}{dt} = kP - H \qquad (H = \text{const.})$$

$$\frac{dP}{kP - H} = dt \Leftarrow \text{Separated}$$

$$\int \quad '' \quad = \int dt = t + c$$

$$\vdots$$

(See Ex. 11, p. 16)

Population with competition

- Const. fraction a reproduces per unit time
- Variable " sP dies " " "

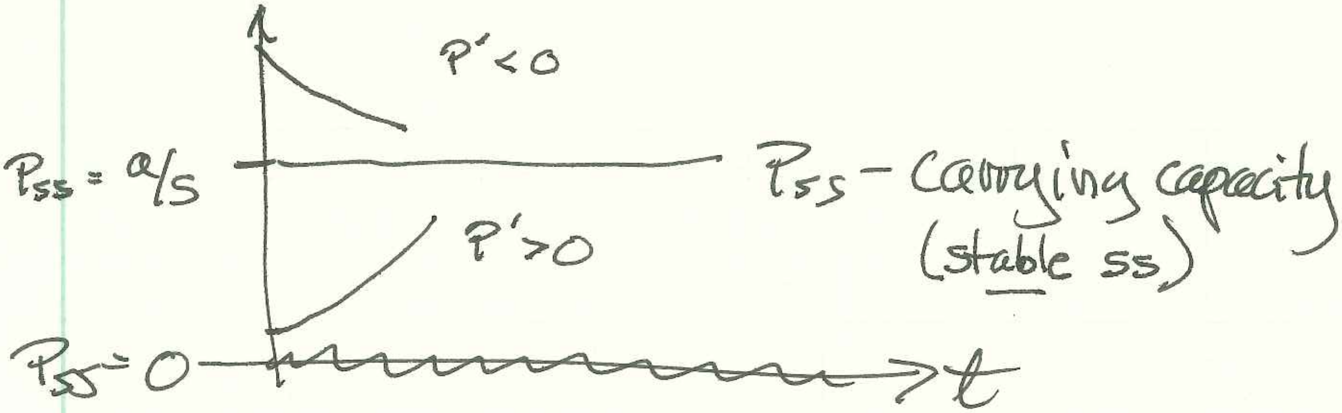
$$P(t+\Delta t) - P(t) = aP(t)\Delta t - (sP)P(t)\Delta t$$

Higher odds of dying if Pop'n large

⋮

$$\frac{dP}{dt} = aP - sP^2 = P(a - sP)$$

$$\Rightarrow 0 = P_{ss}(a - sP_{ss}) \Rightarrow P_{ss} = 0, a/s$$



$$P'(0) = P_i(a - sP_i) = P_i\left[s\underbrace{\left(\frac{a}{s} - P_i\right)}\right]$$

>0 if $\frac{a}{s} > P_i$

<0 " $\frac{a}{s} < P_i$