Test 2 Grading Guide, Sol'ns posted

Mechanics = 70%

Understanding = 30%

Laplace

$$\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s),$$

if integral exists

So far:

- $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$
- $\mathcal{L}\{f'(t)\} = sF(s) - f(0),$

  $F(s) = \mathcal{L}\{f(t)\}$

Use $\mathcal{L}$ to solve IVP (Linear, Const.-coeff.)

1. $\mathcal{L}\{$ entire DE $\}$; use ICs.
   Solve for $Y(s) = \mathcal{L}\{$sol'n of IVP$\}$
2. Invert $Y(s) = \mathcal{L}\{$ " $\}$
   to find sol'n of IVP

Ex: Solve $y' - 4y = 8$, $y(0) = 1$

1. $\mathcal{L}\{$ " $\}$

$$\Rightarrow \mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{8\}$$

$$sY(s) - \overset{=1}{y(0)} - 4Y(s) = 8\mathcal{L}\{1\} = 8\frac{1}{s}$$

$$Y(s)(s-4) - 1 = \frac{8}{s}$$

$$(s-4)Y = \frac{8}{s} + 1$$

$$Y = \frac{8}{s(s-4)} + \frac{1}{s-4} \quad ② \quad ①$$

2. Seek $y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{8}{s(s-4)} + \frac{1}{(s-4)}\right\}$

① $\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} \overset{\#3}{=} \cancel{e^{at} \text{ w/ } a=+4}\ e^{+4t}$

② $\mathcal{L}^{-1}\left\{\dfrac{8}{s(s-4)}\right\}$

$$\frac{8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4} \quad \text{Partial Fractions}$$

$$= \frac{A(s-4) + Bs}{s(s-4)}$$

$$= \frac{(A+B)s - 4A}{s(s-4)}$$

Need $8 = (A+B)s - 4A$

$s^1: \ 0 = A+B \Rightarrow A = -B$

$s^0: \ 8 = -4A \Rightarrow A = -2 \Rightarrow B = 2$

$$\therefore \mathcal{L}^{-1}\left\{\frac{8}{s(s-4)}\right\} = -2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= -2\cdot 1 + 2\cdot e^{4t}$$

So $y(t) = ① + ② = \cancel{4}e^{4t} - 2 + 2e^{4t}$

$\therefore \boxed{y(t) = 3e^{4t} - 2}$ Check ✓

The first step was completed in example 3 of the previous section, where from $\mathcal{L}\left\{y' + ky\right\} = \mathcal{L}\left\{e^{-3t}\right\}$ we found

Transform differential equation and solve for $Y(s)$.

$$Y(s) = \frac{1}{(s+k)(s+3)} + \frac{4}{s+k}.$$

Recall that the initial conditions were incorporated in the transform of the derivative term,

$$\mathcal{L}\left\{y'\right\} = sY(s) - y(0) = sY(s) - 4.$$

The second step was completed in the preceding example by inverting the transform to find

Invert $Y(s)$ to find the solution of the initial-value problem.

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+k)(s+3)} + \frac{4}{s+k}\right\}$$

$$= \frac{(14k-13)e^{-kt} + e^{-3t}}{k-3}, \quad k \neq 0. \ ■$$

2nd-order IVP

■ **EXAMPLE 8** *Solve the initial-value problem*

$$x''(t) + 4x(t) = 0, \quad x(0) = 2, \quad x'(0) = -1.$$

Let $X(s) = \mathcal{L}\{x(t)\}$. Use entry 1 of table 10.2 and the initial conditions to transform the entire differential equation:

Transform differential equation.

$$\begin{aligned} \mathcal{L}\left\{x''(t) + 4x(t)\right\} &= \mathcal{L}\left\{x''(t)\right\} + 4\mathcal{L}\{x(t)\} \\ &= s^2X(s) - sx(0) - x'(0) + 4X(s) \\ &= s^2X - 2s + 1 + 4X \\ &= (s^2+4)X - 2s + 1 = 0. \end{aligned}$$

Solve for X:

Solve for $X(s)$.

$$X(s) = \frac{2s-1}{s^2+4}.$$

Now determine the inverse transform,

Invert $X(s)$.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}.$$

With the aid of entries 5 and 6 in table 10.1, we find

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin 2t.$$

Combining these results gives the solution of the given initial-value problem,

Solution of initial-value problem

$$x(t) = 2\cos 4t - \frac{1}{2}\sin 2t. \ ■$$

Key step in example: $\mathcal{L}^{-1}\left\{ \frac{8}{s(s-4)} \right\}$

$$\frac{8}{s(s-4)} = 8 \cdot \frac{1}{s} \cdot \frac{1}{s-4} \leftarrow \text{Product of trans.}$$

$$= \mathcal{L}\{ ? \}$$

Convolution Thm: $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$

p. 525

$$\Rightarrow F(s)G(s) = \mathcal{L}\left\{ \underbrace{\int_0^t f(r)g(t-r)\,dr}_{\text{CONVOLUTION of } f,\, g} \right\}$$

Ex: $\mathcal{L}^{-1}\left\{ \frac{8}{s(s-4)} \right\}$ via Convolution Thm.

$$= 8 \underbrace{\frac{1}{s}}_{F(s)} \underbrace{\frac{1}{s-4}}_{G(s)}$$

$$F = \frac{1}{s} \Rightarrow f(t) = 1 \qquad G = \frac{1}{s-4} \Rightarrow g(t) = e^{4t}$$

$$8\mathcal{L}^{-1}\{F(s)G(s)\} = 8\int_0^t 1 e^{4(t-r)}\,dr$$

$$= 8e^{4t}\int_0^t e^{-4r}\,dr = 8e^{4t}\left(-\frac{1}{4}e^{-4r}\right)\Big|_{r=0}^{t}$$

$$= -2e^{4t}(e^{-4t}-1) = -2(1-e^{4t})$$

$$= 2e^{4t} - 2 \Leftarrow \text{Same as } ②, \text{ p. } 2\tfrac{1}{2}$$

Shifted transform: $F(s) = \mathcal{L}\{f(t)\} \Rightarrow$

p. 527 $$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Ex: $F(s) = \mathcal{L}\{1\} = \frac{1}{s}$

$$\mathcal{L}\{e^{at}\, 1\} = F(s-a) =$$

Ex. 10 p. 527 Ex: $\mathcal{L}\{e^{-2t} \cos 6t\} =$

See details in book - Example 10, p. 527

Ex. 11, p.527

Ex: $\mathcal{L}^{-1}\left\{ \dfrac{s}{s^2+4s+40} \right\} = ?$

- Factor denom., then partial fractions?

  Roots of $s^2+4s+40=0$

  $$s = \frac{-4 \pm \sqrt{16 - 4\cdot 40}}{2} \leftarrow \underline{\underline{\text{Complex}}}!!$$

p.529 bottom

- Complex roots! Complete square, then shifted trans.

  $$s^2 + 4s + 40 = (s+2)^2 - 2^2 + 40$$

  4/2

  $$= (s+2)^2 + 36$$

See details - ~~middle p. 528~~

- middle of p. 528 to complete square
- inverse trans. top of p. 529

Restatement of bottom of p. 529

Strategy for $\mathcal{L}^{-1}\left\{\frac{\cdots}{s^2+\cdots}\right\}$

- $s^2+\cdots$ has real roots $\Rightarrow$

  Partial fractions: $\frac{\cdots}{s^2+\cdots} = \frac{A}{s-r_1} + \frac{B}{s-r_2}$

  (t func's are $e^{r_1 t}, e^{r_2 t}$)

  or Convolution Thm: $\frac{\sim}{s^2+\cdots} = \frac{\sim}{(s-r_1)(s-r_2)}$

- $s^2+\sim$ has complex roots $\Rightarrow$

  Complete square: $\frac{\sim}{s^2+\sim} = \frac{\sim}{(s-a)^2+k^2}$

  then shifted transform

EXERCISE GUIDE (Continued)

To gain experience . . .	Try exercises
Finding Laplace transforms of functions	13, [illegible]
Solving initial-value problems	14–2[illegible]
With the fundamental ideas of [illegible]	[illegible]

In exercises 1–6,

(i) Write the given transform as an appropriate product and use the convolution theorem to find its inverse.

(ii) Use partial fractions to write the given transform as an appropriate sum and find its inverse.

(iii) Confirm the inverse result using DELAB.

1. $\dfrac{4}{s(s+4)}$

2. $\dfrac{3}{(s-2)(s+3)}$

3. $\dfrac{1}{s^2-16}$

4. $\dfrac{s}{(s+1)(s^2+4)}$

5. $\dfrac{s}{(s-1)(s^2-4)}$

6. $\dfrac{1}{s^2+9}$ (*Hint*: Use $s^2+9=(s+3i)(s-3i)$ and $e^{i3t}=\cos 3t + i\sin 3t$.)

In exercises 7–12, find the inverse of the given transform. Confirm the inverse using DELAB.

7. $\dfrac{6}{s^2-5s+4}$

8. $\dfrac{3s}{2s^2-3s+5}$

9. $\dfrac{s-1}{3s^2-15s+12}$

10. $\dfrac{2s+3}{s^2-2s+2}$

11. $\dfrac{3-2s}{s^2-2s+5}$

12. $\dfrac{2+s}{s^2+4s+1}$

13. Use $\mathcal{L}\left\{e^{at}f(t)\right\}=F(s-a)$ with an appropriate choice of f to find the Laplace transform of each of the following.

(a) $e^{3t}t$

(b) $e^{-4t}\cos \pi t$

(c) $e^{-3t/2}(t^2+4)$

In exercises 14–21,

(i) Working directly from the differential equation, find the transform of the solution of the given initial-value problem.

(ii) Invert that transform to find the solution of the initial-value problem.

(iii) Find the solution of the initial-value problem by another method and verify that the solution obtained by Laplace transforms is correct.

(iv) Use DELAB to confirm both the transform and the solution result.

14. $y'-6y=3$, $y(0)=2$

15. $y'-6y=\sin 3t$, $y(0)=5$

16. $y'+4y=\cos \pi t$, $y(0)=0$

17. $2y'+8y=6e^{-3t}$, $y(0)=-2$

18. $3x''/16+1.41x'+2x=0$, $x(0)=0.75$, $x'(0)=0$ (See example 22 of section 6.4.)

19. $x''+4x=\cos \pi t$, $x(0)=1$, $x'(0)=4$

20. $x''+4x=\sin 2t$, $x(0)=1$, $x'(0)=4$

21. $x''-4x'+4x=0$, $x(0)=2$, $x'(0)=2$

22. Example 7 uses Laplace transforms to solve the initial-value problem

$$y'+ky=e^{-3t}, \quad y(0)=4.$$

It finds

$$y(t)=\frac{(14k-13)e^{-kt}+e^{-3t}}{k-3}.$$

(a) Verify by direct substitution that this solution is correct.

HA 2051-#19

Midterm Course Feedback

Your anonymous, thoughtful feedback on how the course is going so far may help all of us make adjustments that improve your course experience.

Please rank the following, with 1 being the lowest and 5 being the highest

The material is being taught at an appropriate pace	1	2	3	4	5
The professor responds to emails in a timely manner	1	2	3	4	5
Grades for assignments and exams are given in a timely fashion	1	2	3	4	5
The professor encourages in-class participation	1	2	3	4	5
The professor offers help outside of class time	1	2	3	4	5
The professor speaks clearly and audibly	1	2	3	4	5
The professor provides clear and concise explanations	1	2	3	4	5

1. How much time (outside of class) do you spend working in this course?

2. Please identify one or two specific things that you like about this course.

3. Please describe one or two specific things that the instructor(s) could do to improve student learning in this course.