Laplace transforms

For bumps and steps -

- Speed bump at 60 mph
- On-off furnace

Mathematical motivation

Given response $x(t)$ - maybe complicated

Growth/decay? Details?

Oscillations? " ?

⋮

Idea: sample over time t w/ "test func"

e^{-st} - you control s

$$X(s) = \int_{t=0}^{\infty} x(t) e^{-st} \, dt$$

How might work if $x(t) = e^{at}$, say?
(Growth/decay)

$$\overline{X}(s) = \int_{t=0}^{\infty} \left(e^{at}\right) e^{-st}\, dt$$

$$= \int_{t=0}^{\infty} e^{(a-s)t}\, dt$$

$$= \lim_{b\to\infty} \int_{t=0}^{b} e^{(a-s)t}\, dt$$

$$= \lim_{b\to\infty} \left[\left(\frac{1}{a-s} e^{(a-s)t} \right)\Bigg|_{t=0}^{t=b} \right]$$

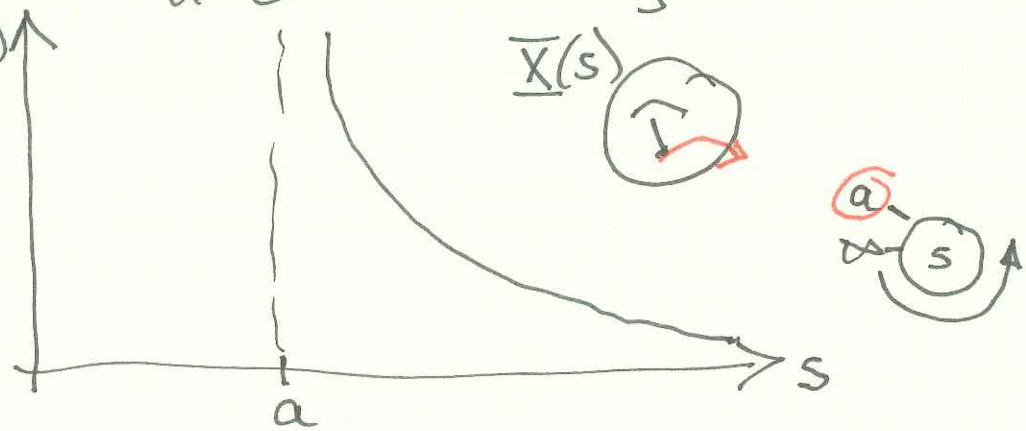
$$= \lim_{b\to\infty} \left[\frac{1}{a-s} e^{(a-s)b} - \frac{1}{a-s} e^{0} \right]$$

$$= \frac{1}{a-s} \lim_{b\to\infty} \left[e^{\underbrace{(a-s)b}_{\text{need }\ominus \Rightarrow a-s<0}} - 1 \right]$$

$a-s<0$

$$= \frac{1}{a-s} \left[0 \quad -1 \right] = \frac{1}{s-a}$$

$\frac{1}{s-a} = \overline{X}(s)$



Defn: Laplace Transform

$$\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$$

Valid for values of s for which $\int_0^\infty \cdots dt$ exists.

Ex: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\ , \quad s > a$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt$$

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Laplace changes $\frac{d}{dt}$ to multiplication by s!

Calculus $\rightarrow$ algebra

$\Rightarrow$ Solve IVPs

Ex: $\mathcal{L}\{f'(t)\} = s\underbrace{F(s)}_{=\mathcal{L}\{f(t)\}} - f(0)$; $F(s) = \mathcal{L}\{f\}$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= \lim_{b\to\infty}\left[\int_{t=0}^{b} f'(t)\, e^{-st}\, dt\right]$$

[[Parts: $\int u\, dv = uv - \int v\, du$: $dv = f'\,dt$, $u = e^{-st}$

$v = \int dv = f$; $du = -s\, e^{-st}\, dt$]]

$$= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_{t=0}^{b} - \underbrace{\int_{t=0}^{b} f(t)(-s)\, e^{-st}\, dt}_{\rightarrow \mathcal{L}\{f(t)\}}\right]$$

$$= \lim_{b\to\infty}\Big[e^{-sb} f(b) - e^{0} f(0) + s\int_{t=0}^{b} f(t)\, e^{-st}\, dt\Big]$$

$$= 0 - f(0) + s\,\mathcal{L}\{f\} \checkmark$$

Laplace transform pairs

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\dfrac{1}{s}$
$n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}, s > a$
$\sin at$	$\dfrac{a}{s^2+a^2}, s > 0$
$\cos at$	$\dfrac{s}{s^2+a^2}, s > 0$
$\sinh at$	$\dfrac{a}{s^2-a^2}, s > a$
$\cosh at$	$\dfrac{s}{s^2-a^2}, s > a$
$\delta(t-a)$	$e^{-as}, a > 0$

Properties of Laplace Transforms

$$F(s) = \mathcal{L}f(t), \quad G(s) = \mathcal{L}g(t)$$

Transform of a derivative:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Convolution theorem:

$$\mathcal{L}\left\{\int_0^t f(r)g(t-r)\,dr\right\} = F(s)G(s)$$

Shifted transform:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Transform of a periodic function of period T:

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Derivative of a transform:

$$\mathcal{L}\{-tf(t)\} = F'(s)$$
$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s), \quad n = 1, 2, \ldots$$

Transform of an integral:

$$\mathcal{L}\left\{\int_0^t f(r)\,dr\right\} = \frac{F(s)}{s}$$

Integral of a transform:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(r)\,dr$$

Transform of a shifted function using unit step function $\mathcal{U}$:

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s), \quad a > 0$$

In exercises 1–5,

(i) Use the definition (1) to find the Laplace transform of each of the following functions. Give the values of s for which each transform is defined.

(ii) Use the linearity of the Laplace transform and the appropriate entries in table 10.1 to verify that the transform you obtained from the definition is correct. Indicate explicitly those steps which depend upon the linearity of the Laplace transform. Identify by number each entry from table 10.1 that you use.

(iii) Compare each result with the transform expression given by DELAB.

1. $4 - 9e^{-4t}$
2. $3t^2$ (*Hint*: Integrate by parts.)
3. $7 + \pi$
4. $\cos \pi t$ (*Hint*: Integrate by parts twice as in example 4.)
5. $2t - 5$

In exercises 6–13, use the linearity of the Laplace transform and the appropriate entries in table 10.1 to find the Laplace transform of each of the following functions. Indicate explicitly those steps which depend upon the linearity of the Laplace transform. Identify by number each entry from table 10.1 that you use. Compare each result with the transform expression given by DELAB.

6. 14
7. $2t^3$
8. $14 - 2t^3$
9. $t + 2e^{2t}$
10. $5t^3 + 3 \sin 2t$
11. $A \sin \omega t$, A, ω constants
12. $2 \sin 4t - 9 \cosh 6t$
13. $\sum_{n=1}^{4} nt^n$

In exercises 14–17, assume that the Laplace transform of the solution of each initial-value problem exists.

(i) Find the transform of the solution of the initial-value problem working directly from the differential equation.

(ii) Solve each initial-value problem by a method of your choice. Compute the transform of the solution you find and show that it is the same as the one you obtained directly from the differential equation in part (i). Confirm both the solution and the transform using DELAB.

14. $y' - 4y = \sin 2t$, $y(0) = 3$
15. $2y' + 7y = 5t + e^{-2t}$, $y(0) = -1$
16. $x' - 2x = t^2 - 3\cos \pi t$, $x(0) = 3$
17. $y' - ky = \alpha e^{-t}$, $y(0) = y_i$, k, α constants

In exercises 18–25, use the definition of Laplace transform to derive the indicated entry from table 10.1.

18. Entry 1; of what other entries is this a special case?
19. Entry 2 for $n = 1$ $\mathcal{L}\{1\} = \frac{1}{s}$
20. Entry 2 for $n = 2$
21. Entry 2 for n an arbitrary nonnegative integer (Use mathematical induction.)
22. Entry 5
23. Entry 6 (*Hint*: Use the definition of the hyperbolic sine function, entry 3 in table 10.1, and the linearity of the Laplace transform.)
24. Entry 7 (See the preceding hint.)
25. Find $\mathcal{L}\{\sinh at + \cosh at\}$ using entry 3.

In exercises 26–27, use the indicated entry from table 10.2 to find the Laplace transform of the given function. If the transform can be evaluated by a second method, do so to verify the accuracy of your answer. If a transform does not exist, explain why.

26. Use $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ with an appropriate choice of f to find the Laplace transform of
 (a) $4e^{4t}$
 (b) $3t^2$
 (c) $-a \cos at$
27. Use $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ with an appropriate choice of f to find the Laplace transform of
 (a) $-a^2 \sin at$
 (b) $a^2 e^{at}$
 (c) $6t$

In exercises 28–29, use the definition of Laplace transform to derive the indicated entry in table 10.2. In each case, $F(s) = \mathcal{L}\{f(t)\}$, and all transforms are assumed to exist.

28. $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ (*Hint*: Integrate the definition of Laplace transform by parts using $dv = f'(t)\,dt$.)
29. $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ (*Hint*: Apply the result of exercise 26 to the function $g'(t)$, where $g(t) = f'(t)$.)

In exercises 30–31, use the definition of Laplace transform to find the transform of the given function. Sketch the graph of each function before you attempt to find its transform.