Undetermined coeff's

- Particular sol'n of

  Const. Coeff. Linear Nonhomo DE

$$\text{RHS} = \begin{cases} \text{poly} \\ \text{exponential} \\ \sin/\cos \end{cases}$$

- Guess y_p "looks like" RHS

Detail

If guess solves homo. DE, mult. guess by t

" new " " " " " " " "

with forcing terms $f(x)$ from the same family.

The essential idea of undetermined coefficients is to assume a trial particular solution that involves the same sorts of functions as appear in the forcing term. For example, if the forcing term is

$$\sin 2\pi t,$$

then the trial particular solution is

$$A\cos 2\pi t + B\sin 2\pi t.$$

The coefficients A and B are determined by substituti... the trial solution into the differential equation.

If part of a trial particular solution is a solution of the *homogeneous*
the ...on, then the coefficient of that part will disappear when it is substituted
4.4 for illus...ntial equation. Hence, its coefficient will remain undetermined.
In this situatio...ns, this difficulty can be circumvented by multiplying
efficients for first-order equ...dependent variable. See the examples of section
trial solution of a second-order equ...
leave a solution of the homogeneous equatio...ence between undetermined co-
we simply multiply by the independent variable on... equations. Multiplying a
The rules for efficiently guessing a form of the partic... variable may yet
be more elegant, constructing a trial solution) are summarized in ...that case,
includes several natural extensions of the first-order guidelines of, e.g., forc...
terms which are the product of a polynomial and an exponential.

TABLE 6.3 Particular solutions via undetermined coefficients for the constant-coefficient, second-order linear equation $b_2y'' + b_1y' + b_0y = f(x)$

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$b_2y'' + b_1y' + b_0y = f(x)$

Forcing Term $f(x)$	Trial Particular Solution
$a_nx^n + \cdots + a_1x + a_0$	$A_nx^n + \cdots + A_1x + A_0$
$(a_nx^n + \cdots + a_1x + a_0)e^{qx}$	$(A_nx^n + \cdots + A_1x + A_0)e^{qx}$
$(a_nx^n + \cdots + a_1x + a_0)\cos px$ $+ (b_nx^n + \cdots + b_1x + b_0)\sin px$	$(A_nx^n + \cdots + A_1x + A_0)\cos px$ $+ (B_nx^n + \cdots + B_1x + B_0)\sin px$
$ae^{qx}\cos px + be^{qx}\sin px$	$Ae^{qx}\cos px + Be^{qx}\sin px$

If the assumed form of the particular solution solves the corresponding homogeneous equation, multiply the assumed form by x. Repeat if necessary.

Lowercase letters a, a_i, b, b_i are constants given in the forcing function. The coefficients to be determined are denoted by uppercase letters A, A_i, B, B_i.

with that of problem

$2x$

because the forcing
ying an exponential
ion of the homoge-
y this entire expres-
trial solution

$-2x$

3 because no terms

e first trial solution
:ause it contains no

G_0,

roblem 5 is a poly-
olution satisfy the

particular solution

s $y_{p1}, \ldots, y_{p5}$ are
ummed. We leave

ı for the general
:. The value of
superposition) is
: of the drudgery
ts.

6.6.3 Exercises

EXERCISE GUIDE

To gain experience ...	Try exercises
Forming trial particular solutions	1–27, 32(a–b), 33(b), 35
Finding values of undetermined coefficients	1–27, 35
Determining when to use undetermined coefficients	28
Finding general solutions	1–27, 29–31, 39–40
Solving initial-value problems	32(a–b), 33(a)
With the foundations of undetermined coefficients	36–38, 41
Using superposition	34–35
Analyzing solution behavior	32(c), 33(b)

27

In exercises 1–~~10~~, (1-10)

(i) Give an efficient form of a trial particular solution.

(ii) Find a particular solution.

(iii) Find a general solution.

(iv) Compare your results with those of DELAB.

1. $y'' - 9y = 14 - 2x$
2. $y'' - 9y = x^2 - 1$
3. $y'' - 9y = e^{-2x}$
4. $y'' - 9y = xe^{-2x}$
5. $y'' - 9y = 4 - e^{-3x}$
6. $y'' - 9y = xe^{-3x}$
7. $y'' - 9y = 2e^{3x} - e^{-3x} - 5x$
8. $4x''(t) + 8x'(t) + 5x(t) = e^t(\sin t/2 - 6)$
9. $4x''(t) + 8x'(t) + 5x(t) = 6\sin t/2$
10. $4x''(t) + 8x'(t) + 5x(t) = t - 2$
11. $w''(x) + 9w(x) = x\sin 3x + 2\cos 4x$
12. $w''(x) + 9w(x) = e^{3x}$
13. $w''(x) + 9w(x) = xe^{-x}$
14. $w''(x) + 9w(x) = x^2 - 3$
15. $w''(x) + 9w(x) = 2 - \cos 3x$
16. $2z''(t) - 7z'(t) + 3z(t) = e^{x/2}\sin \pi x$
17. $2z''(t) - 7z'(t) + 3z(t) = 6e^{3t}$
18. $2z''(t) - 7z'(t) + 3z(t) = -1$

1. $y_p = A$

2. $y_p = (Be^{-3x})x$

19. $2z''(t) - 7z'(t) + 3z(t) = e^{x/2}\sin \pi x + 6e^{3t} - 1$
20. $y'' + 4y' + 40y = xe^{-2x}\cos 6x$
21. $y'' + 4y' + 40y = xe^{-2x}$
22. $y'' + 4y' + 40y = -\sin 6x$
23. $x'' + x' - 6x = -3t$
24. $x'' + x' - 6x = te^{-3t} - \cos 2t$
25. $9y'' + 36y' + 4y = xe^{-2x/3}$
26. $4y''(x) + 8y'(x) + 5y(x) = \cos x/2 + x^3$
27. $w'' + w' - 6w = e^{2x} - e^{-3x}$
28. Verify that
$$x''(t) = -g$$
is indeed a constant-coefficient, linear, second-order equation, as the text claims in example 36.
29. Verify that the solution
$$x_g(t) = -gt^2/2 + C_1 t + C_2$$
of
$$x''(t) = -g$$
obtained in example 36 is indeed a *general* solution.
30. Verify that the solution
$$y_g(x) = 4x^2e^{-2x} + C_1e^{-2x} + C_2xe^{-2x}$$
of
$$y'' + 4y' + 4y = 8e^{-2x}$$
obtained in example 37 is indeed a *general* solution.

Ex: $y'' + y' = 7$

① Guess: $y_p = A$

② Plug in: $A'' + A' \stackrel{?}{=} 7$

$0 \neq 7$

$y = \text{const.}$ solves Homo DE

③ New guess: $y_p = tA$

$y_p'' + y_p' = (tA)'' + (tA)'$

$= 0 + A \stackrel{?}{=} 7$

$\Rightarrow A = 7$

$\therefore y_p = 7t$

$y_g = 7t + C_1 y_1 + C_2 y_2$

$= 7t + \underbrace{C_1 + C_2 e^{-t}}_{r = 0, -1}$

Ex: Part. Sol'n of $x'' + 9x = 2\sin 4t$

1st Homo: $x'' + 9x = 0$ (S-M)

Guess: $x_h = e^{rt}$

$\Rightarrow r^2 \cancel{e^{rt}} + 9\cancel{e^{rt}} = 0 \Rightarrow r^2 = -9 \Rightarrow r = \pm 3i$

$(\alpha = 0, \beta = 3)$

$x_h = C_1 \sin 3t + C_2 \cos 3t$

Part: Guess: $x_p = A \sin 4t + B\cos 4t$

$x_p'' = -16A \sin 4t - 16B \cos 4t$

$\therefore x_p'' + 9x_p = -\underbrace{\overset{7}{\cancel{15}}A}_{}\sin 4t - \underbrace{\overset{7}{\cancel{18}}B}_{}\cos 4t$

$= 2\sin 4t + 0$

$\Rightarrow A = -\frac{2}{7}, \quad B = 0$

$\therefore x_g = -\frac{2}{7}\sin 4t + C_1 \sin 3t + C_2 \cos 3t$

Ex: Part. sol'n of $x'' + 9x = 2\sin 3t$

Part: ① $x_p = A\sin 3t + B\cos 3t$

$x_p'' + 9x_p = 0 \Leftarrow$ (p.2)

② $x_p = \underbrace{t}_{\to\infty}\underbrace{(A\sin 3t + B\cos 3t)}_{\text{oscillate}}$

31. Verify that the solution

$$y_g(x) = x^3e^{-2x} + C_1e^{-2x} + C_2xe^{-2x}$$

of

$$y'' + 4y' + 4y = 6xe^{-2x}$$

obtained in example 38 is indeed a *general* solution. If you completed the previous exercise, can you use some of its analysis here? Explain. $x'' + 9x = 2\sin\begin{bmatrix}4\\3\end{bmatrix}t$

32. Consider the undamped, forced spring-mass system model

$$mx''(t) + kx(t) = A\sin\omega t, \quad x(0) = x_i, \quad x'(0) = v_i,$$

where A, ω are fixed parameters. (Recall that m is mass, k is the spring constant, and x_i, v_i are the initial position and velocity.)

(a) Assume $\omega \neq \sqrt{k/m}$ and solve the initial-value problem.

(b) Assume $\omega = \sqrt{k/m}$ and solve the initial-value problem. RESONANCE

(c) How does the behavior of the solutions obtained in parts (a) and (b) differ? How does the value of ω affect the solution procedure you use?

33. (a) Solve the damped, forced spring-mass system model

$$mx''(t) + px'(t) + kx(t) = A\sin\omega t,$$
$$x(0) = x_i, \quad x'(0) = v_i,$$

where A, ω are fixed parameters. (Recall that m is mass, p is the damping or friction coefficient, k is the spring constant, and x_i, v_i are the initial position and velocity.)

(b) The previous exercise considered the undamped ($p = 0$) version of this model, and two cases arose, $\omega = \sqrt{k/m}$ and $\omega \neq \sqrt{k/m}$. Do such considerations arise here? Does the solution of this problem remain bounded for all time?

34. Example 39 obtained the particular solution

$$y_p(x) = (3x^3 - x^2)e^{-2x} + \frac{4\pi}{(4+\pi^2)^2}\cos\pi x - \frac{4-\pi^2}{(4+\pi^2)^2}\sin\pi x$$

of

$$y'' + 4y' + 4y = 18xe^{-2x} - \sin\pi x - 2e^{-2x}$$

by decomposing the original problem into two subproblems:

Problem 1: $y'' + 4y' + 4y = 18xe^{-2x} - 2e^{-2x}$,

Problem 2: $y'' + 4y' + 4y = -\sin\pi x$.

Verify by direct substitution that y_p, which is a sum of a particular solution of problem 1 and a particular solution of problem 2, is indeed a solution of the full problem. Which part of y_p is a solution of problem 1? Of problem 2? Do these parts each yield the expected forcing terms when they are substituted into the left side of the equation? Show how your calculations support your answer to this last question.

35. Example 42 broke the problem of finding the form of a particular solution of

$$y'' - y' - 6y = 3x\sin 2x - 5\cos 3x + x^3(e^{-2x} + e^{2x}) + \pi - x^4$$

into a series of subproblems:

Problem 1: $y'' - y' - 6y = 3x\sin 2x$

Problem 2: $y'' - y' - 6y = -5\cos 3x$

Problem 3: $y'' - y' - 6y = x^3e^{-2x}$

Problem 4: $y'' - y' - 6y = x^3e^{2x}$

Problem 5: $y'' - y' - 6y = \pi - x^4$

Find a particular solution of the original differential equation by finding a particular solution of each of these subproblems and summing. You may use as much of the information in example 42 as you find useful.

36. In its search for a particular solution of

$$x''(t) = -g,$$

example 36 proposed three forms for a trial solution of $x'' = -g$. They are

$$x_{p1}(t) = A_0,$$
$$x_{p2}(t) = A_0t,$$
$$x_{p3}(t) = A_0t^2.$$

Confirm the wisdom of finally using x_{p3} by direct substitution of the first two trial solutions in $x''(t) = -g$. Explain why these two trial solutions are unsatisfactory. Show that the third trial solution is not a solution of the homogeneous equation $x'' = 0$.

37. In its search for a particular solution of

$$y'' + 4y' + 4y = 8e^{-2x},$$

example 37 proposed three forms for a trial solution of $y'' + 4y' + 4y = 8e^{-2x}$:

$$y_{p1}(x) = A_0e^{-2x},$$
$$y_{p2}(x) = A_0xe^{-2x},$$
$$y_{p3}(x) = A_0x^2e^{-2x}.$$

Confirm the wisd
tion of y_{p1} and y
why these two t
that x_{p3} is not a
$y'' + 4y' + 4y =$

38. In its search for a

y

example 38 prop
$y'' + 4y' + 4y =$

$y_{p1}($

$y_{p2}($

$y_{p3}($

Confirm the wisd
tution of the first
$6xe^{-2x}$. Explain v
factory. Show tha
of the homogeneo

39. Example 39 found

$y'' + 4y' +$

is

$y_p(x) = (3x$

Find a general solu

6.7 ■ ANALY

Forced spring-mass n