Const. coeff., Linear, Homo., 2^{nd}-order

$$b_2 y'' + b_1 y' + b_0 y = 0$$

Solve for 2 Lin. Ind. (why?) (Gen'l Sol'n!)

Charac. eqn: guess $\Rightarrow y = e^{rx}$

$$y' = r e^{rx}$$

$$y'' = r^2 e^{rx}$$

Plug-in: $b_2 r^2 \cancel{e^{rx}} + b_1 r \cancel{e^{rx}} + b_0 \cancel{e^{rx}} = 0$

CE: $\boxed{b_2 r^2 + b_1 r + b_0 = 0}$

Quadratic eqn $\underbrace{b_2}_{a} r^2 + \underbrace{b_1}_{b} r + \underbrace{b_0}_{c} = 0$

has roots

$b_1^2 - 4b_2b_0$		
>0	real	1. $r_1 \neq r_2$
$=0$	real	2. $r_1 = r_2 = r$
<0	complex conjugate	3. $r_1, r_2 = \alpha \pm i\beta$

$$r = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2b_0}}{2b_2}$$

Linearly Independent sol'ns of

$$b_2 y'' + b_1 y' + b_0 y = 0$$

p. 260 1. $r_1 \neq r_2 \Rightarrow y_1 = e^{r_1 x}, \; y_2 = e^{r_2 x}$

p. 262 2. $r_1 = r_2 = r \Rightarrow y_1 = e^{rx}, \; y_2 = x e^{rx}$

p. 268 3. $r = \alpha \pm i\beta \Rightarrow y_1 = e^{\alpha x} \cos \beta x$

$$y_2 = e^{\alpha x} \sin \beta x$$

[Key: Euler: $e^{(\alpha \pm i\beta)x} = e^{\alpha x} e^{\pm i \beta x}$

$= e^{\alpha x} (\cos \beta x \pm i \sin \beta x)$,

$i = \sqrt{-1}$]

$$y'' - 4y' + 4y = 0 \Rightarrow (r-2)^2 = 0$$

$$\left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} - 2\right) y = 0$$

$$\underbrace{\qquad\qquad e^{2x}}_{=0}$$

$$(\quad)\underbrace{\qquad x e^{2x} \qquad}_{= e^{2x}}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{=0}$$

6.3.4 Exercises

EXERCISE GUIDE

To gain experience ...	Try exercises
Using the characteristic equation method	1–17, 23
Finding equations, given solutions	11–17
Verifying linear independence of solutions	1–10, 18–19, 20–21(b)
Verifying given functions are solutions	20–22(a)
Finding general solutions	1–10, 18–19, 20–21(c), 23
Solving initial-value problems	1–10
With the foundations of the characteristic equation method	11–17, 22(b), 23–24
Understanding repeated roots	22, 24–26

$e^{rx} \to$ Quad. Eqn.

In exercises 1–10,

(i) Find the characteristic equation of the given differential equation.

r values

(ii) Find two linearly independent solutions of the differential equation.

(iii) Verify that these solutions are linearly independent for all values of the independent variable.

(iv) Write a general solution of the differential equation.

(v) Solve the differential equation subject to the given initial conditions.

$C_1y_1 + C_2y_2$

(vi) Confirm the general solution you obtain using the analytic solution tool in DELAB.

$r = 2, -1$

1. $2y'' - 2y' - 4y = 0$, $y(0) = 1$, $y'(0) = -4$
2. $y'' = y$, $y(0) = 0$, $y'(0) = 2$
3. $y'' = -2y' + y$, $y(0) = 1$, $y'(0) = -1$
4. $x''(t) + x'(t) - 6x(t) = 0$, $x(1) = 6$, $x'(1) = 0$
5. $2w''(x) - 3w'(x) + w(x) = 0$, $w(0) = 2$, $w'(0) = 1$
6. $8z(t) - 10z'(t) - 3z''(t) = 0$, $z(0) = 2$, $z'(0) = 1$
7. $4y'' - 8y' + 4y = 0$, $y(0) = -3$, $y'(0) = 4$
8. $x''/3 - 2x' - 9x = 0$, $x(0) = 0$, $x'(0) = 1$
9. $12u'' + 5u' - 2u = 0$, $u(1) = 1$, $u'(1) = -1$
10. $y'' + 2ay' + a^2y = 0$, $y(0) = m$, $y'(0) = n$, a, m, n constants

In exercises 11–17, write a constant-coefficient, second-order differential equation that has the given pair of functions as solutions. Write the most general equation you can.

11. e^x, e^{-x} $\Rightarrow (r-1)(r-(-1)) = 0$
12. e^{2x}, e^{-3x}
13. e^x, xe^x
14. $-e^{-x}$, xe^{-x}
15. e^{x+1}, e^{2x-1}
16. e^x, π
17. e^x
18. Verify by direct substitution that

$$y_1 = e^{-4x}, \quad y_2 = e^{3x}$$

are solutions of the differential equation

$$y'' + y' - 12y = 0$$

of example 15. Show that y_1, y_2 are linearly independent. Construct a general solution of this differential equation.

19. If the characteristic equation

$$b_2r^2 + b_1r + b_0 = 0$$

of the general constant-coefficient, homogeneous equation

$$b_2y'' + b_1y' + b_0y = 0$$

has the distinct real roots $r_1, r_2, r_1 \neq r_2$, then the differential equation has two solutions,

$$y_1 = e^{r_1x}, \quad y_2 = e^{r_2x}.$$

6.3.4/# 1(ii)

Verify e^{2x}, e^{-x} are LI

$$(\ )' = 0' \Rightarrow \begin{cases} C_1 e^{2x} + C_2 e^{-x} = 0 \text{ all } x \Rightarrow C_1 = C_2 = 0 \\ 2C_1 e^{2x} - C_2 e^{-x} = 0 \end{cases}$$

Solve C_1, C_2

In exercises 1–16,

6.4.4

(i) Find the characteristic equation of the given differential equation. If the concept of characteristic equation is not appropriate, explain why and go on to the next exercise.

(ii) Find two linearly independent solutions of the differential equation.

(iii) Verify that these solutions are linearly independent for all values of the independent variable.

(iv) Write a general solution of the differential equation.

(v) Confirm the general solution you obtain using the analytic solution tool in DELAB.

1. $y'' - 9y = 0$

2. $4x''(t) + 8x'(t) + 5x(t) = 0$

3. $w''(x) + 9w(x) = 0$ $r^2+9=0 \Rightarrow r=\pm 3i$

4. $2z''(t) - 7z'(t) + 3z(t) = 0$ $w'' = -9w$

5. $y'' - 4y' + 40y = 0$

6. $y'' - 4y' + 40y = \cos \pi x$

7. $x'' + x' - 6x = 0$

8. $9y'' + 36y' + 4y = 0$

9. $9y'' + 36y' + 4y^2 = 0$

10. $mx''(t) + kx(t) = 0$, m, k constants

11. $mx''(t) + kx(t) = k[h(t) - h(0)]$, m, k constants

12. $L\theta'' + g\theta = 0$, L, g constants

13. $L\theta'' + g \sin\theta = 0$, L, g constants

14. $mx'' + px' + kx = 0$, m, p, k constants, $p^2 - 4mk > 0$

15. $mx'' + px' + kx = 0$, m, p, k constants, $p^2 - 4mk = 0$

16. $mx'' + px' + kx = 0$, m, p, k constants, $p^2 - 4mk < 0$

In exercises 17–23,

(i) Write a general solution of the differential equation.

(ii) Solve the initial-value problem.

17. $u'' - 4u' + 40u = 0$, $u(\pi/6) = 1$, $u'(\pi/6) = -1$

18. $9x'' + 36x' + 4x = 0$, $x(0) = 2$, $x'(0) = 3$

19. $4y'' + 8y' + 5y = 0$, $y(\pi) = 0$, $y'(\pi) = -4$

20. $z'' + z' - 6z = 0$, $z(0) = 4$, $z'(0) = -6$

21. $x'' + 9x = 0$, $x(\pi) = 3$, $x'(\pi) = -9$

22. $2w'' - 7w' + 3w = 0$, $w(0) = 8$, $w'(0) = 12$

23. $x'' - 9x = 0$, $x(0) = 18$, $x'(0) = 18$

24. In the undamped spring-mass system model derived in section 5.1,

$$mx''(t) + kx(t) = 0, \quad x(0) = x_i, \quad x'(0) = v_i,$$

the parameters m, k in the differential equation are positive constants.

(a) Find a general solution of the differential equation.

(b) Argue that your solution is purely oscillatory, neither growing nor decaying with time.

(c) Find the angular frequency and period of these oscillations.

(d) Argue that the period of the oscillations in this spring-mass system depends only on the mass m and the spring constant k, not upon the initial conditions; i.e., the period of the oscillations is independent of how the motion is initiated.

(e) Find the solution of this initial-value problem.

(f) Find the maximum displacement of the mass from its equilibrium position. Does it reduce to the value you expect when the initial velocity v_i is zero?

25. In the damped spring-mass system model derived in section 5.1,

$$mx''(t) + px'(t) + kx(t) = 0,$$
$$x(0) = x_i, \quad x'(0) = v_i,$$

the parameters m, p, k in the differential equation are positive constants.

(a) Find a general solution of this differential equation. If you need to consider different ranges of the parameters m, p, k, state them clearly.

(b) Argue that your solution is decaying toward a steady state of zero, regardless of the values of m, p, k. Which parameter(s) in the equation determine(s) the rate of decay? Are they the one(s) you expect?

(c) For what ranges of the parameters does the solution of this equation exhibit oscillations? Are the underlying oscillations of the same frequency as those of the undamped spring-mass model? (See exercise 24.)

(d) Find the solution of the initial-value problem.

26. The first example in this section begins with two complex exponential solutions

$$z_1 = e^{(-4+2i)x}, \quad z_2 = e^{(-4-2i)x}$$

of the differential equation

$$y'' + 8y' + 20y = 0.$$

It then obtains two new solutions

$$y_1 = e^{-4x} \cos 2x, \quad y_2 = e^{-4x} \sin 2x$$

as linear combinations of z_1, z_2.

(a) Write y_1 and y_2 as linear combinations of z_1, z_2.

(b) Verify by substitution that y_1, y_2 are indeed solutions of $y'' + 8y' + 20y = 0$. Why is this direct verification actually unnecessary?

27. The last example in this section shows that the differential equation

$$\tfrac{3}{16}x''(t) + 2x(t) = 0,$$

from the undamped spring-mass model of example 1, section 5.1, has the characteristic equation

$$\tfrac{3}{16}r^2 + 2 = 0.$$

The roots of this equation are $r = \pm\sqrt{32/3}\,i \approx \pm 3.27i$. Consequently, $3x''/16 + 2x = 0$ has the linearly independent solutions

$$z_1 = e^{3.27it}, \quad z_2 = e^{-3.27it}.$$

Paul Davis **MA2051B13**

WeBWorK assignment number PWDSolveOrder2 is due : 11/20/2013 at 09:00pm EST.

The link

http://users.wpi.edu/ pwdavis/Courses/MA2051B13/MA2051B13syllabus.htm

leads to the syllabus for the course. It contains the homework and test schedule, grading policy, and other information.

The primary purpose of WeBWorK is to let you know that you are getting the correct answer or to alert you if you are making some kind of mistake. Usually you can attempt a problem as many times as you want before the due date. However, if you are having trouble figuring out your error, you should consult the book, or ask a fellow student, one of the TA's or your professor for help. Don't spend a lot of time guessing – it's not very efficient or effective.

Give 4 or 5 significant digits for (floating point) numerical answers. For most problems when entering numerical answers, you can if you wish enter elementary expressions such as $2 \wedge 3$ instead of 8, $sin(3*pi/2)$instead of -1, $e \wedge (ln(2))$ instead of 2, $(2+tan(3))*(4-sin(5)) \wedge 6-7/8$ instead of 27620.3413, etc. Here's the **list of the functions** which WeBWorK understands.

You can use the Feedback button on each problem page to send e-mail to the professors.

---

1. (1 pt) Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + y' = 4x$$

Answer: $y(x) =$ ________ $+C_1$ ________ $+C_2$ ________.

NOTE: The order of your answers is important in this problem. For example, webwork may expect the answer "A+B" but the answer you give is "B+A". Both answers are correct but webwork will only accept the former.

Answer(s) submitted:

-
-
-

(incorrect)

---

2. (1 pt) Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = 2e^{-x}$$

Answer: $y(x) =$ ________ $+C_1$ ________ $+C_2$ ________.

NOTE: The order of your answers is important in this problem. For example, webwork may expect the answer "A+B" but the answer you give is "B+A". Both answers are correct but webwork will only accept the former.

Answer(s) submitted:

-
-
-

(incorrect)

---

3. (1 pt) Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 4y = 4x$$

Answer: $y(x) =$ ________ $+C_1$ ________ $+C_2$ ________.

NOTE: The order of your answers is important in this problem. For example, webwork may expect the answer "A+B" but the answer you give is "B+A". Both answers are correct but webwork will only accept the former.

Answer(s) submitted:

-
-
-

(incorrect)

---

4. (1 pt) Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = 2\sin(x)$$

Answer: $y(x) =$ ________ $+C_1$ ________ $+C_2$ ________.

NOTE: The order of your answers is important in this problem. For example, webwork may expect the answer "A+B" but the answer you give is "B+A". Both answers are correct but webwork will only accept the former.

Answer(s) submitted:

-
-
-

(incorrect)

with forcing terms $f(x)$ from the same family.

The essential idea of undetermined coefficients is to assume a trial particular solution that involves the same sorts of functions as appear in the forcing term. For example, if the forcing term is

$$\sin 2\pi t,$$

then the trial particular solution is

$$A\cos 2\pi t + B\sin 2\pi t.$$

The coefficients A and B are determined by substituting the trial solution into the differential equation.

If part of a trial particular solution is a solution of the *homogeneous*
the ...on, then the coefficient of that part will disappear when it is substituted
4.4 for illus...ential equation. Hence, its coefficient will remain undetermined.
In this situati...ns, this difficulty can be circumvented by multiplying
efficients for first-order equa... dependent variable. See the examples of section
trial solution of a second-order equ...
leave a solution of the homogeneous equatio...ence between undetermined co-
we simply multiply by the independent variable o... equations. Multiplying a
The rules for efficiently guessing a form of the partic... variable may yet
be more elegant, constructing a trial solution) are summarized in ...that case,
includes several natural extensions of the first-order guidelines of, e.g., forc...
terms which are the product of a polynomial and an exponential.

TABLE 6.3 Particular solutions via undetermined coefficients for the constant-coefficient, second-order linear equation $b_2y'' + b_1y' + b_0y = f(x)$

$b_2y'' + b_1y' + b_0y = f(x)$	
Forcing Term $f(x)$	**Trial Particular Solution**
$a_nx^n + \cdots + a_1x + a_0$	$A_nx^n + \cdots + A_1x + A_0$
$(a_nx^n + \cdots + a_1x + a_0)e^{qx}$	$(A_nx^n + \cdots + A_1x + A_0)e^{qx}$
$(a_nx^n + \cdots + a_1x + a_0)\cos px$	$(A_nx^n + \cdots + A_1x + A_0)\cos px$
$+ (b_nx^n + \cdots + b_1x + b_0)\sin px$	$+ (B_nx^n + \cdots + B_1x + B_0)\sin px$
$ae^{qx}\cos px + be^{qx}\sin px$	$Ae^{qx}\cos px + Be^{qx}\sin px$

If the assumed form of the particular solution solves the corresponding homogeneous equation, multiply the assumed form by x. Repeat if necessary.

Lowercase letters a, a_i, b, b_i are constants given in the forcing function. The coefficients to be determined are denoted by uppercase letters A, A_i, B, B_i.