Vectors

Introduction

These notes describe everything that needs to be said for this course about vectors, and a bit more. You may find parts of these notes to be familiar. You should realize that these notes are concise. Other than the introductory remarks about J. W. Gibbs, each of the remarks in these notes is potentially useful someplace in Freshman Physics.

The *Vector* description of mathematical quantities was substantially the creation of the late Josiah Willard Gibbs, Professor of Mathematical Physics at Yale University in the late 19th century. Until Gibbs' work, physicists described all quantities with numbers or, at considerable expense in effort, with *quaternions*. Gibbs found a better method, namely the use of vectors. Gibbs also took the laws of thermodynamics, which at the time described how steam engines work, and applied them to chemistry, creating by deduction from first principles the field of *chemical thermodynamics*. At age 60 he published *Elementary Principles in Statistical Mechanics*, which showed that Newton's laws of motion and a new law of nature applied to point atoms in a box predict the laws of thermodynamics, thus creating in modern form the science of *statistical mechanics*. For creating a new and highly useful piece of mathematics, and setting down the basics of two branches of science, Gibbs is reasonably identified as being the greatest American physicist of the second millennium. However, his vector ideas were meant to help his engineering and physics students, most of whom were not as bright or well prepared as you all are.

Coordinate Systems

To discuss vectors, we first have a few remarks about coordinate systems.

Physics is concerned with describing where objects are in space, and when events happened. To describe where objects were located, we introduce coordinate systems. The simplest coordinate system is the Cartesian or (x, y, z)-coordinate system. The coordinates are distances that we move, parallel to the x, y, and z axes, to get from the origin to the point of interest. In this system the coordinate axes are straight, perpendicular lines. The simplest cartesian coordinate system is displayed on a sheet of graph paper, in which points are labelled by their x and y coordinates. For physics, a sheet of graph paper is inadequate. Graph paper is two-dimensional, but space is three dimensional. To specify where a point is located in space, we need to specify three coordinates (x, y, z).

Because objects move, at different times an object will have been in different places. We therefore also need to specify the time t at which an object was at a particular point in space. The statement 'specify the time' is not nearly so simple as it sounds, because time does not have the properties that you were taught by implication in grade school, when you were asked "what time is it?".

The statement that a coordinate system is 'cartesian' does not identify it uniquely. There are many different cartesian coordinate systems. For example, two coordinate systems could differ from each other because their origins are not in the same place. Two coordinate systems could differ from each other because they are rotated with respect to each other, so that, e.g., the x axis of one system is pointed parallel to the y-axis of the other coordinate system. Finally, two coordinate systems might differ because they are moving with respect to each other. As an example of moving coordinate systems, consider a section of the Turnpike along which a truck is being driven. The mileposts along a straight section of road define a set of x-coordinates, with some point on the road as the origin. A second coordinate system might take the tailgate of a truck as its origin, a coordinate axis being painted on the side of the truck, with the truck being driven down the road at constant speed. In the first coordinate system, the truck is moving: It keeps passing mileposts. In the second coordinate system, the truck is stationary: Its tailgate remains fixed at $(0, 0, 0)$.

You should also be aware of circular polar coordinates, in which the coordinates x and y are replaced by a distance r from the z-axis and an angle θ. There are an extremely wide range of mathematical coordinate systems. All coordinate systems *[Well, not quite all: All coordinate systems that are related to each other in a continuous way, so that pairs of points that have nearby mathematical values of their coordinates in one coordinate system have nearby mathematical values of their coordinates in the other coordinate system, which includes all coordinate systems that almost any of you have ever heard of.]* have an important feature in common: They all have three coordinate axes, and assign to each point in space three (not two or four) coordinates. *Those of you who have been exposed to a bit of real analysis will be aware that the number of points in a line, and the number of points in a plane, are equal to each other, which says some unexpected things about functions.*

Vectors and Their Components

What is a vector? The most limited definition is that a vector is a quantity with a direction and a length or *magnitude*. Vectors appear on every weather report, because the wind "speed"– e.g., 15 mph from the West–is a vector. More generally, a vector is a quantity that is fully specified by a list of its *components*. Most generally, a vector is an ordered list of numbers $a_1, a_2, a_3, \ldots$, each a_i being a number, with the subscripts 1, 2, 3, being values of the *index* i for the list.

We distinguish a vector from a scalar, a scalar being a quantity that is completely described by a single number. For example, on the same weather report, the air temperature is a scalar.

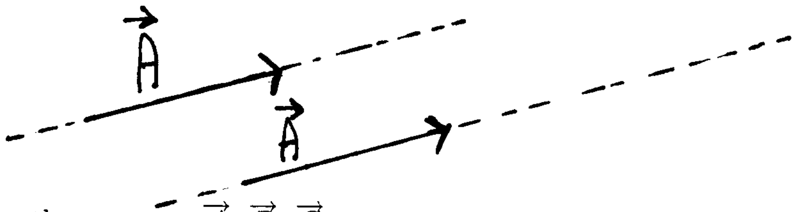
Vectors are fundamental for descriptions of modern physics. On one hand, vectors are a great labor-saving tool, letting you write an equation while using far fewer characters than would otherwise be needed. On the other hand, vector equations are more general than the corresponding scalar equations, because a vector equation remains true when you change your coordinate system.

Let's start with some notation. Most textbooks denote a vector by printing it in boldface **a** or by putting an arrow over it $\overrightarrow{a}$. The same vector is also written $\underset{\sim}{a}$, the tilde *under* the letter being the author's instruction to the typesetters to set a letter in boldface. On the blackboard, I will usually write $\vec{a}$. Each vector has a *magnitude*–for a position vector, the magnitude is the vector's length. The magnitude of a vector $\overrightarrow{a}$ is written a, | **a** |, or || **a** ||.

Vectors are physical quantities, so they have dimensions and are expressed in physical units.

While vectors have a direction and a magnitude, they do not have a location. Two parallel vectors having the same magnitude are the same vector, even if they reference different points in space. A 15 knot west wind is the same wind in San Francisco and in Boston. There is, however, an object known as a *vector field*, in which each point in space has a vector assigned to it. Weather maps showing wind speed and direction at each city are showing a vector field. In a vector field, the location of the vector matters: if the 10 knot breeze here and the 200 knot breeze on top of Mount Washington have their locations interchanged, there will be physical consequences. The magnetic field lines drawn as surrounding a magnet are also a vector field.

We draw a vector as an arrow, the ends of the arrow being the tail and the head. A vector pointing out of the plane toward the reader is represented as a circle, sometimes with a dot in the middle representing the arrowhead. A vector pointing into the plane away from the reader is represented by a cross or 'X', the X representing the arrow's tail feathers. Two vectors are parallel if their arrows lie along parallel lines.



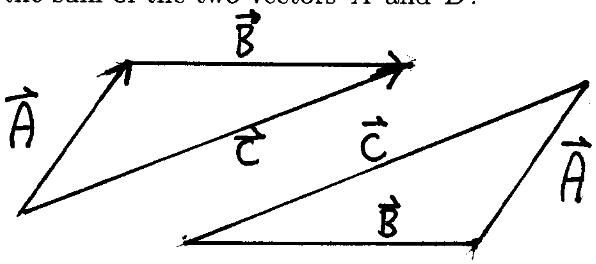
For three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, we may write a vector equation

$$\vec{A} = \vec{B}, \tag{1}$$

which means that the two vectors $\vec{A}$ and $\vec{B}$ have the same magnitude and the same direction. We write

$$\vec{A} + \vec{B} = \vec{C} \tag{2}$$

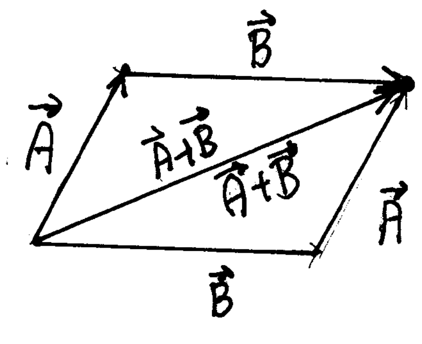
to describe the sum of the two vectors $\vec{A}$ and $\vec{B}$.



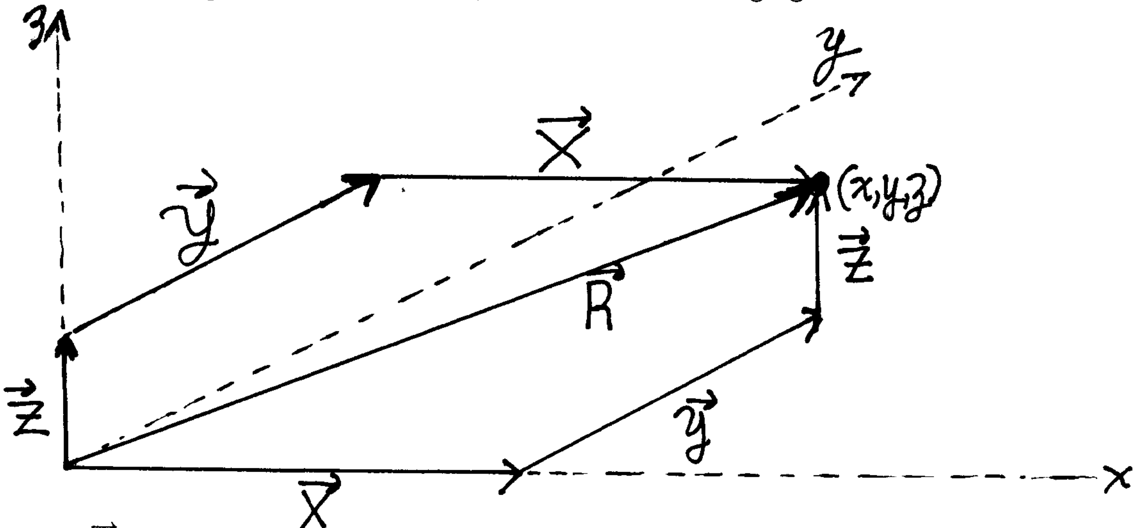
A vector sum is described geometrically by either of the two figures above. To add two vectors, we drag them around until the head of one vector and the tail of the other vector are coincident. The sum vector closes the triangle as shown. [There are some mathematical complications that arise when we say that we can move vectors around without changing which way they point. These complications do not arise in this course.] Vector addition is *commutative*, meaning that

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}. \tag{3}$$

Geometrically, eq 3 is the statement that a parallelogram is a closed figure, whose two halves share a common diagonal, as seen in the next figure.



As a particular example of vector addition, consider the following figure.



The vector $\vec{R}$ starts at the origin and goes out to a point in space. For each vector, there is a corresponding point, and vice versa, so we give the point at the end of the vector the name $\vec{R}$. The vector $\vec{R}$ can be written as the sum of a vector $\vec{X}$ parallel to the x axis, a vector $\vec{Y}$ parallel to the y axis, and a vector $\vec{Z}$ parallel to the z axis. Because vector addition is commutative, there are several ways to order that sum, but each sum uses the same $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ to generate $\vec{R}$.

It is also possible to multiply a vector by a scalar. Multiplying a vector by a scalar creates a new vector. The new vector points in the same direction as the old vector, up to a sign, but (unless the scalar is unity) has a different length. Scalar multiplication of vectors is a *signed* operation. If you multiply a vector by a negative number, the vector switches its direction by 180 degrees. The product $s\vec{A}$ is a vector, whose magnitude is $\mid s \mid A$ and whose direction is parallel or antiparallel to $\vec{A}$.

A particularly important vector is the *unit vector*, denoted $\hat{A}$. The unit vector is defined by

$$\hat{A} = \left(\frac{1}{A}\right)\vec{A}. \tag{4}$$

A is the magnitude of $\vec{A}$, so the unit vector is in the same direction as $\vec{A}$. However, $\hat{A}$ and $\vec{A}$ in general have different magnitudes. By convention, the dimensions of a vector and the dimensions of its magnitude are the same, so a unit vector has dimensions *unity.* Every vector has a corresponding unit vector. The unit vector is a perfectly ordinary vector in all respects. It has a direction and a magnitude. However, the magnitude of a unit vector is 1. Indeed, any vector may be written as the product of its length and direction as

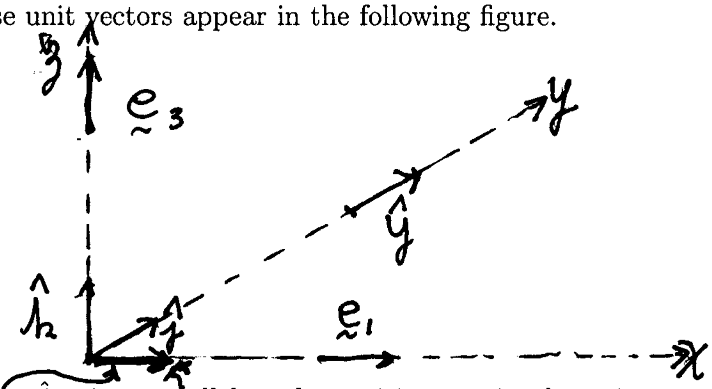
$$\vec{A} = A\hat{A}. \tag{5}$$

While every vector has its corresponding unit vector, there is a particular set of unit vectors that are so important that they are given their own symbols. This unit vectors are called the *basis vectors.* Space is three-dimensional, and correspondingly there are three basis vectors. In Cartesian coordinates, the three basis vectors are

$$\hat{\imath} \equiv \hat{x} \equiv \mathbf{e}_1 \tag{6}$$

$$\hat{\jmath} \equiv \hat{y} \equiv \mathbf{e}_2 \tag{7}$$
$$\hat{k} \equiv \hat{z} \equiv \mathbf{e}_3 \tag{8}$$

Examples of these unit vectors appear in the following figure.



The unit vector $\hat{\imath}$ points parallel to the positive x-axis; the unit vector $\hat{\jmath}$ points parallel to the positive y-axis; the unit vector $\hat{k}$ points parallel to the positive z-axis. I will generally refer to the three coordinate unit vectors as $(\hat{\imath}, \hat{\jmath}, \hat{k})$, but the other notations are in common use, so you should be able to recognize them.

In terms of Figure 3, we also have

$$\vec{X} = X\hat{\imath} \tag{9}$$
$$\vec{Y} = Y\hat{\jmath} \tag{10}$$
$$\vec{Z} = Z\hat{k} \tag{11}$$

which leads to

$$\vec{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k} \tag{12}$$

Equation 12 is both general and fundamental. For the vector $\vec{R}$, the quantities X, Y, and Z are, respectively, the x, y, and z *components* of the vector $\vec{R}$. The components of a vector are **scalars**. The x-component of the vector R is X and **not** $X\hat{\imath}$. As an alternative way to write the same vector, consider the notation

$$\vec{R} = (X, Y, Z). \tag{13}$$

In the above (X, Y, Z) is an *ordered* list of three numbers. What do we mean by an ordered list? Not only does it matter which numbers are in the list, but the order of the numbers is significant.In general, $(X, Y, Z) \neq (Y, X, Z)$, unless $X = Y$.

Scalar multiplication of a vector may be written in terms of the vector components. Scalar multiplication has the effect

$$s(X, Y, Z) = (sX, sY, sZ) \tag{14}$$

The components of a vector $\vec{A}$ may also be labelled by subscripts, namely

$$\vec{A} = A_x\hat{\imath} + A_y\hat{y} + A_z\hat{k} \equiv (A_x, A_y, A_z), \tag{15}$$

so that A_i is the i^{th} component of $\vec{A}$. Observe that (A_x, A_y, A_z) is a second way to write a vector – a second *notation* for expressing vectors – namely to write the vector you write the list of its coordinates in a coordinate system. In eq. 15, the two ways to write a vector are *isomorphic*. The two ways to write a vector are different, but any statement that is true or false when you write vectors using one notation for a vector is exactly as true or false when you write vectors using the other notation.

The components of a vector clarify the arithmetical interpretation of several of the above equations. In particular, a vector equation

$$\vec{A} + \vec{B} = \vec{C} \tag{16}$$

is simply a symbol for three component equations. Equation 16 means neither more nor less than the three equations

$$A_x + B_x = C_x \tag{17}$$
$$A_y + B_y = C_y \tag{18}$$
$$A_z + B_z = C_z \tag{19}$$

the above being scalar equations that involve only standard mathematical processes on scalars, e. g., $2 + 2 = 4$. Whenever it is necessary to interpret a vector equation, the vector equation can always be replaced with three scalar equations, and conventional arithmetic operations can then be applied to the x, y, and z components. Correspondingly, if you are ever unsure what a vector equation means, once you write the vectors out in terms of their components you will end up with three equations, each equation containing only simple scalars, which you should be able to evaluate.

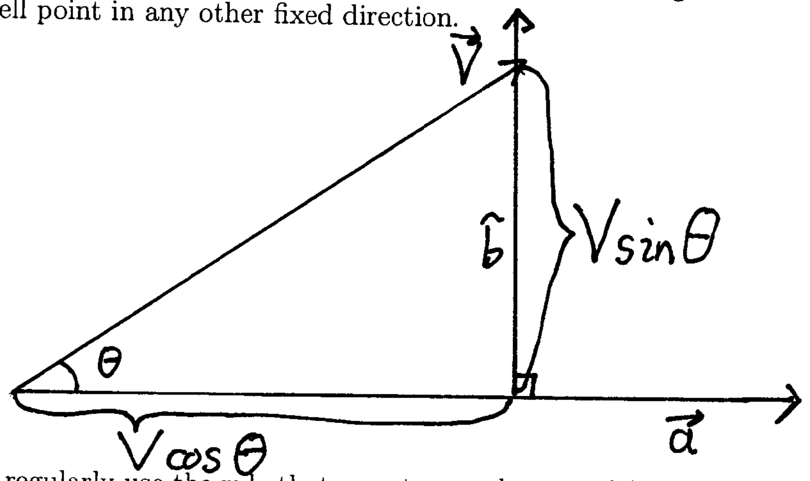
The magnitude of a vector can be computed from its components, namely

$$A = \left[A_x^2 + A_y^2 + A_z^2\right]^{1/2}. \tag{20}$$

Equation 20 is the Theorem of Pythagoras in three dimensions. (Unlike many other equations above, this definition of the magnitude requires that you use Cartesian coordinates.)

Vector Projections

The directions and components of a vector are related to the projections of that vector onto different axes. What do we mean by the *projection* of a vector? The projection of vector $\vec{V}$ onto an axis **a** is shown in the next Figure. Here **a** could point along the x or y or z direction, but it could as well point in any other fixed direction.

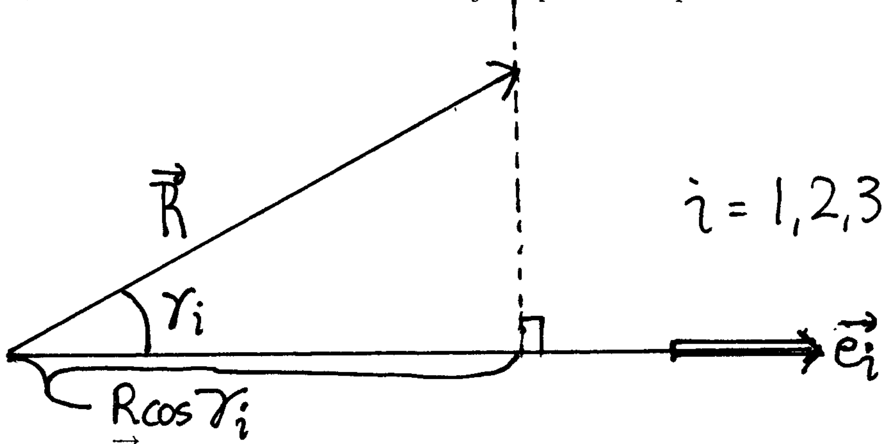


We will regularly use the rule that a vector can be moved from point to point without changing its length or direction, so that we can always juxtapose the tails of $\vec{V}$ and **a**. These two vectors via their three points (two heads, a common tail) define a plane. In the Figure, the vector **b** is perpendicular to **a**, and with complete generality lies in the $\vec{V} - \mathbf{a}$ plane. The angle between $\vec{V}$ and **a** is θ. The projections of $\vec{V}$ onto **a** and **b** are scalars, namely $V\cos(\theta)$ and $V\sin(\theta)$, respectively. From the theorem of Pythagoras, the magnitude V of $\vec{V}$ is

$$V = \left[V^2\cos^2(\theta) + V^2\sin^2(\theta)\right]^{1/2}, \tag{21}$$

which may be recognized as the standard trigonometric addition theorem $\sin^2(\theta) + \cos^2(\theta) = 1$.

For the vector $\vec{R}$ of Figure 3, it is interesting to make projections onto the x, y, and z axes. Figure 5 is an appropriate representation for each projection, because for each projection of $\vec{R}$ onto a coordinate axis, $\vec{R}$ and the coordinate axis can be juxtaposed in a plane as shown in the Figure.



The angles between $\vec{R}$ and the x, y, and z axes are by convention γ_1, γ_2, and γ_3, respectively. The cosines of these three angles are by convention termed the *direction cosines*. The components of $\vec{R}$ are obtained via projection as

$$R_x = R\cos(\gamma_1) \tag{22}$$
$$R_y = R\cos(\gamma_2) \tag{23}$$
$$R_z = R\cos(\gamma_3) \tag{24}$$

Substituting the above into eq 20, one finds

$$\cos^2(\gamma_1) + \cos^2(\gamma_2) + \cos^2(\gamma_3) = 1, \tag{25}$$

which is the standard solid-geometry addition rule for the direction angles. Equation 25 has an important implication, namely that the three angles γ_1, γ_2, and γ_3 are not independent. If you know any two of them, you know the third. Therefore, in order to specify the direction of a vector, you only need to specify two angles, not three angles. You have all known for a very long time an example of specifying a direction with two angles. Suppose you want to specify the direction from the center of the Earth to a point on the Earth's surface. One way to do that is to specify the location on the surface. However, you can specify a location on the surface of the earth with two angles, namely the longitude and the latitude, so you can specify a direction from the center of the earth by specifying the longitude and latitude at which the direction crosses the surface of the earth.

As an exercise, confirm that the three Cartesian basis vectors have components $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

Multiplication of Vectors

In addition to addition, subtraction, and projection of vectors, it is also possibly to multiply two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. There are three different products, three different ways to multiply two vectors together. The products differ in the outcome of the multiplication process; each product has several names. The *scalar* or *dot* product $\overrightarrow{A} \cdot \overrightarrow{B}$ yields a scalar (a single number). The *vector*, *cross*, or *Grassmann* product $\overrightarrow{A} \times \overrightarrow{B}$ yields a vector. The *outer* or *dyadic* product, variously written $\overrightarrow{A} \otimes \overrightarrow{B}$ or simply $\overrightarrow{A}\overrightarrow{B}$ yields a tensor (here, a 3x3 matrix).

In writing the various vector products, we first note the *Curie Principle.* The Curie principle states that in any equation $a = b$ the left and right side of the equation must have the same vector nature. a and b may both be scalars, they may both be vectors, or they may both be tensors, but they must be the same. An equation that equates two mathematical objects having different vector natures, e.g., $\overrightarrow{A} = 7$, cannot possibly be right, because it does not make mathematical sense. You can see this by comparison with eqs 18-19. If you try to plug the erroneous form $\overrightarrow{A} = 7$ into eqs 18-19, the three components of $\overrightarrow{A}$ fit appropriately into the left sides of the three equations 18-19, but the right side of $\overrightarrow{A} = 7$ has no components to fit into the right sides of eqs 18-19.

As a special case, the equation $\overrightarrow{A} = \overrightarrow{0}$ is usually written as $\overrightarrow{A} = 0$. The arrow over the 0 is by convention omitted, but it is there; in this last result, 0 actually stands for $0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$.

The following gives a variety of expressions for vector products, in terms of the Cartesian components of the vectors. Clearly there is more than one way to transform a pair of vectors, and their six components, into a number or another vector. The products we show here are important because they are independent of the choice of Cartesian coordinates. What is meant by independent? If we were to rotate the coordinates, so that the new x, y, and z axes were not parallel to the old x, y, and z axes, the components A_x, A_y,... of the two vectors would change. For the particular vector products discussed here, those changes all cancel, in the sense that the values of the scalar and vector products of two vectors have the same magnitude and direction when the coordinate system is rotated.

The scalar product is defined in terms of its Cartesian components to be

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y + A_z B_z. \tag{26}$$

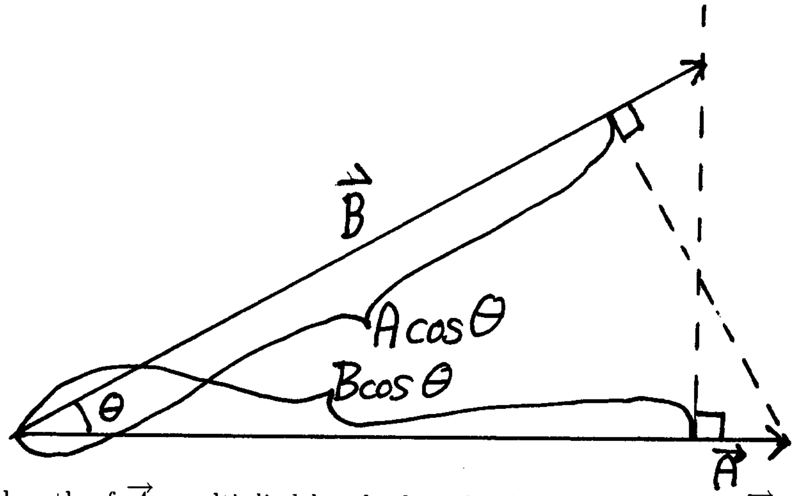
If $\overrightarrow{A} = \overrightarrow{B}$, $\overrightarrow{A} \cdot \overrightarrow{B} \equiv \overrightarrow{A} \cdot \overrightarrow{A} = A^2$ is the square of the length of $\overrightarrow{A}$. The scalar product is both commutative ($\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A}$) and distributive($\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C}$).

It may also be shown from equation 26 that

$$\overrightarrow{A} \cdot \overrightarrow{B} = AB\cos(\theta), \tag{27}$$

θ being the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$. (Try this for yourself–it's not trivial without finding the trick.)

As seen in the Figure



$AB\cos(\theta)$ is geometrically the length of $\overrightarrow{A}$, multiplied by the length of the projection of $\overrightarrow{B}$ onto $\overrightarrow{A}$. The dot product is symmetric between $\overrightarrow{A}$ and $\overrightarrow{B}$, so we might equally say that $AB\cos(\theta)$ is geometrically the length of $\overrightarrow{B}$, multiplied by the length of the projection of $\overrightarrow{A}$ onto $\overrightarrow{B}$.

The components of a vector are the projections of that vector onto the basis vectors, namely

$$A_x = \overrightarrow{A} \cdot \hat{\imath} \tag{28}$$
$$A_y = \overrightarrow{A} \cdot \hat{\jmath} \tag{29}$$
$$A_z = \overrightarrow{A} \cdot \hat{k} \tag{30}$$

so that any vector can be written in terms of the basis vectors as

$$\overrightarrow{A} = (\overrightarrow{A} \cdot \hat{\imath})\hat{\imath} + (\overrightarrow{A} \cdot \hat{\jmath})\hat{\jmath} + (\overrightarrow{A} \cdot \hat{k})\hat{k} \equiv \sum_{i=1}^{3}(\overrightarrow{A} \cdot \mathbf{e}_i)\mathbf{e}_i. \tag{31}$$

where for once I used the $\mathbf{e}_i$ notation for the three basis vectors. Exercise: Confirm these two equations by writing $\overrightarrow{A}$ and the unit vectors in (x, y, z) notation, and carrying out the scalar products.

If you forgot one of the basis vectors, the set of basis vectors would be termed *incomplete.* If you tried to write eq 31 while omitting, say, $\hat{\imath}$, the resulting equation would in general be wrong, because the part of $\overrightarrow{A}$ lying along the x-axis would be missing on the right hand side.

The vector or cross product $\overrightarrow{A} \times \overrightarrow{B}$ of two vectors creates a third vector. Writing $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{C}$, the cross-product vector $\overrightarrow{C}$ is perpendicular to both $\overrightarrow{A}$ and $\overrightarrow{B}$. Recalling that the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ define a plane, $\overrightarrow{C}$ is perpendicular to that plane.

The direction of the cross product vector is generated via the *right hand rule.* To apply this rule, the vector $\overrightarrow{A}$ is identified with the thumb, the thumb being rotated in the plane of the palm until it is perpendicular to the arm. The vector $\overrightarrow{B}$ is lined up with the forefinger, which is pointed more or less straight out from the hand, parallel to the arm. The cross product vector $\overrightarrow{C}$is represented by the middle finger, which is rotated until it is perpendicular to the plane of the palm. For example, if $\overrightarrow{A}$ is to your left, your thumb would point left, and if $\overrightarrow{B}$ is straight away from you, your index finger would point away from you. Inevitably, unless you have a great deal of joint flexibility or have broken something, if you have followed the above instructions, while using your right hand, your middle finger is pointing straight down.

The vector cross product is distributive, so specifying the cross products of the unit vectors completely specifies the cross product. However, the crossproduct is anticommutative, meaning

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, so the list of specifications is

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \tag{32}$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath} \tag{33}$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath} \tag{34}$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \tag{35}$$
$$\hat{k} \times \hat{\jmath} = -\hat{\imath} \tag{36}$$
$$\hat{\imath} \times \hat{k} = -\hat{\jmath} \tag{37}$$

This list of equations looks difficult to remember, but fortunately there is a mnemonic. To recover all these equations, list the letters of the basis vectors twice in a row *ijkijk*. The leftward normal reading direction is positive, the other direction is negative. All of the equations 32-37 can then be read from the list. For example in *ijkijk* if one starts at the first k and reads rightwards one has *kij*, corresponding to eq 34. If one starts at the first k and instead reads leftwards (gaining from the rule a minus sign), one has *-kji*, corresponding to eq 36.

The vector crossproduct can also be written entirely in terms of components, namely

$$\vec{A} \times \vec{B} = (A_x B_y - A_y B_x)\hat{k} + (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} \tag{38}$$

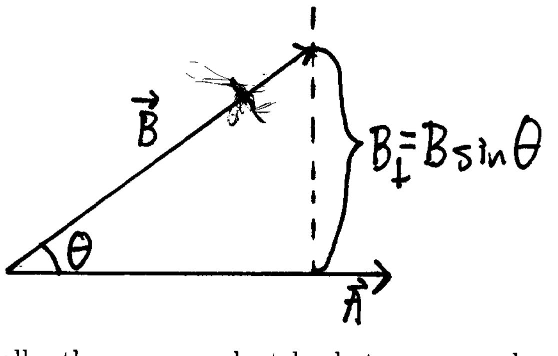
If two vectors are parallel, their cross product is $\vec{A} \times \vec{B} = \vec{0}$. To show this, note that if $\vec{A}$ and $\vec{B}$ are parallel that $\vec{A} = s\vec{B}$, s being a scalar, and substitute in equation 38.

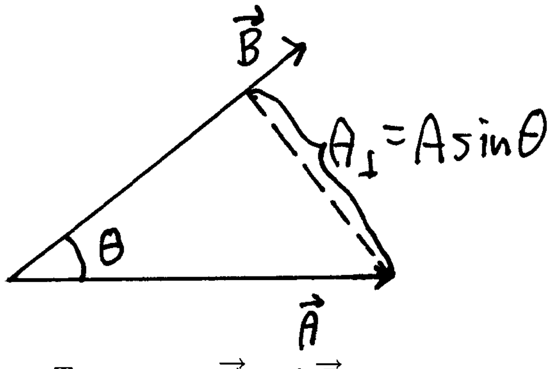
The magnitude of the crossproduct is

$$\|\vec{A} \times \vec{B}\| = \|\vec{A}\| \; \|\vec{B}\| \; \sin(\theta). \tag{39}$$

Here θ is the angle created between the two vectors $\vec{A}$ and $\vec{B}$ by juxtaposing their tails. This equation may be derived by beginning with $(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B})$ and doing component expansions and trigonometry.

The cross product and its magnitude have not one but two geometric representations. First, for $\|\vec{A} \times \vec{B}\|$ the constructions in Figures a and b show that $\|\vec{A}\| \; \|\vec{B}\| sin(\theta)$ is the product of the length of one vector, multiplied by the length of the projection, of the other vector, onto the perpendicular to the first vector. One sometimes sees the perpendicular component $B_\perp$ of a vector defined by $\|\vec{B}\| \sin(\theta) \equiv B_\perp$. Using this definition, $\|\vec{A} \times \vec{B}\| = AB_\perp = A_\perp B$. Note that the definition of $B_\perp$ implicitly requires that you specify the other vector $\vec{A}$ from which the perpendicular is constructed. Also, note that $B_\perp$ involves a distance perpendicular to $\vec{A}$, not to $\vec{B}$. The torque laboratories make a systematic use of perpendicular components.





Finally, the cross product leads to areas and volumes. Two vectors $\vec{A}$ and $\vec{B}$ can be used to define a parallelogram, the two vectors being two adjacent sides of the parallelogram. The quantity $\|\vec{A} \times \vec{B}\|$ is the area of this parallelogram. Similarly, three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, if their tails are brought together, define a parallelopiped–a six-sided solid whose opposite faces are parallel. The volume of the parallelopiped is $V = \|(\vec{A} \times \vec{B}) \cdot \vec{C}\|$. Observe that the triple product only makes mathematical sense if your evaluate first the vector product and then the scalar product, so the clarifying parentheses surrounding $(\vec{A} \times \vec{B})$ are usually omitted.

Note that up to an overall sign the vectors can be placed in that series in any order, implying but not proving the identity $\vec{A} \times \vec{B} \cdot \vec{C} = \vec{A} \cdot \vec{B} \times \vec{C}$. This identity may be confirmed by expanding all products in terms of vector components. [Do this as an Exercise. Hint: If you have never learned how to use a computer algebra program, you may now want to see why they can remove pain.]

Finally, the tensor or dyadic product $\mathbf{M} = \vec{A} \otimes \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is calculated component by component as

$$\mathbf{M}_{ij} = A_i B_j. \tag{40}$$

M_{ij} is a 3×3 matrix; it has nine components. M_{ij} is also a tensor; it has two indices, i and j, which independently can be 1, 2, or 3. The dyadic product is not invariant under a change of coordinates, but several of its properties are invariants. The simplest invariant of M is its *trace.* Tr(M) is the sum of the diagonal components, namely

$$\mathrm{T}r(\mathbf{M}) = \sum_{i=1}^{3} M_{ii} \tag{41}$$

As an exercise, confirm $\mathrm{Tr}(\vec{A} \otimes \vec{A}) = \vec{A} \cdot \vec{A} = A^2$. We will not make extensive use of dyadic products in this course.

Appendix

The above discussion has said several things that while true are incomplete. This Appendix is written for students likely to get an A or better in the course. It points out a few incomplete statements, so that you will not make too many unwarranted assumptions about how the world works.

We mentioned above the existence of complete and incomplete sets of basis vectors. It is also possible to define an *overcomplete* set of basis vectors, for example by adding to $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ an additional basis vector $\hat{u} = \hat{\imath} + \hat{\jmath}$. Superficially, $\hat{u}$ does not look very useful: If you tried to write equation 31 using all four unit vectors, you would not get an equality, because part of $\vec{A}$ would be counted twice. It is possible, if you are careful, to write each vector in terms of the four unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, and $\hat{u}$. However, because this set of four basis vectors is overcomplete, there is more than one set of four components x, y, z, u that correctly describes each vector $\mathbf{r}$, so that you cannot tell if two vectors are equal by comparing their x, y, z, u components. It turns out that by using an overcomplete set of basis vectors and abandoning equation 31 you can gain useful flexibility in real applications, as seen in *wavelet* descriptions of images and signals and in quaternion descriptions of rotations.

The preceding discussion referred only to Cartesian (x-x-z) coordinates. Cartesian coordinates have some peculiar properties that other coordinate systems lack. In particular, if you take any of the Cartesian basis vectors, e.g., $\hat{\imath}$, and translate it parallel to itself, it is still a basis vector.

As a contrast, consider cylindrical coordinates, in which the three axes are r, θ, and z. Everywhere in space the unit vector $\hat{r}$ is perpendicular to the z-axis, so that at a point on the x-axis

$\hat{r} = \pm\hat{\imath}$ while at a point on the y axis $\hat{r} = \pm\hat{\jmath}$. That is, if you translate $\hat{r}$ parallel to itself from the x-axis to the y-axis, $\hat{r}$ ceases to be the radial unit vector. Correspondingly, if you compare $\hat{r}$ at two locations, you find that the direction of $\hat{r}$ is a function of position.

In conventional Cartesian coordinates, the length of a vector can be written in terms of its components as $R^2 = x^2 + y^2 + z^2$. It is sometimes convenient to rewrite the length as

$$R^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} r_i g_{ij} r_j \tag{42}$$

In this equation, $r_1 \equiv x$, $r_2 \equiv y$, and $r_3 \equiv z$. The object g_{ij} is the *metric tensor*, a 3x3 matrix, namely

$$g_{ij} \equiv \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}. \tag{43}$$

In special relativity, the two endpoints of a vector displacement each have Cartesian $(x,\ y,\ z)$ locations; they also have the times at which the locations were measured. A displacement vector then has four components, namely x, y, z, and t, the four components giving the difference between the two endpoints along the three Cartesian axes, and the difference in times between measuring the two endpoints. In special relativity, the length of a vector becomes

$$R^2 = \sum_{i=1}^{4}\sum_{j=1}^{4} r_i g_{ij} r_j, \tag{44}$$

with the extra definition $r_4 \equiv t$ and a g_{ij} given by

$$g_{ij} \equiv \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{Bmatrix}. \tag{45}$$

where c is the speed of light.

An important feature of the length of a vector is that it is an *invariant*, that is, observers in different reference frames (for example, reference frames that are moving with respect to each other) agree as to how long a vector is. *Invariance* turns out to have some involved and surprising implications, which are treated in the theory of special relativity.

Getting equation 45 to work properly when comparing accelerating reference frames is much more difficult that getting equation 43 to work while comparing accelerating reference frames. It is challenging to find a definition of the length that agrees with equation 45 and that simultaneously works for measurements done in reference frames that are accelerating with respect to each other. It turns out that writing this definition, and working out its physical implications, is much more difficult than might be expected. The core results were deduced by Albert Einstein as the Theory of General Relativity. One physical consequence of this effect is that the rate at which a clock runs depends on the local value of the gravitational potential. Many of you have seen a practical consequence of this: A GPS system for determining location depends on measuring times very accurately. If the satellites that drive GPS systems did not include general relativity corrections to their clocks, they would soon tell you that you were miles away from your actual location.

(Correspondingly, if Einstein and successors had not noticed General Relativity, its existence would have become inescapable so soon as satellite location systems were launched, because the satellite systems would mysteriously have failed to work.)

Finally, we have regularly referred to vectors as being parallel, without worrying very hard about what 'parallel' means. The question of parallelism is not at all trivial. Suppose we specify a vector by specifying two points that are infinitesimally far apart, separated by a displacement in the direction of the vector, the magnitude of the vector being recorded separately. Suppose we displace this vector infinitesimally by displacing its tail and head points by equal amounts that are small relative to the length. Larger displacements are obtained by repeating the small displacement process many times. If we take the vector around in a triangle or simple curve back to its starting point, do we get back the same vector? In simple plane geometry, the answer is 'Yes'. In the real world, thanks to general relativity, the answer is 'No'. In the real world, the angle between the new and old vectors is determined by General Relativity and the mass enclosed by the triangle. If you put something like a star (the Sun is adequate) inside a triangle, the sum of the interior angles of the triangle differs appreciably from π radians. This phenomenon has been observed experimentally.