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How many bits are needed to transmit a qubit reliably?

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Abstract

Ways of encoding the state of a qubit in a classical message of x bits are investigated, based on regular and semi-regular divisions of the sphere. The fidelities of the various schemes are computed and compared against each other. Transmission of the global phase of a qubit is also considered. © 1998 Elsevier Science B.V.

A qubit is any two-state quantum system that can be prepared in an arbitrary superposition of its basis states. If $|0\rangle$ and $|1\rangle$ are an orthonormal pair of basis states, the most general state of the qubit can be represented as $|\Psi(\theta,\phi)\rangle = \cos(\theta/2)|0\rangle + e^{-i\phi}\sin(\theta/2)|1\rangle$ where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. A qubit is the basic unit of information storage in the quantum computer of the future. Qubits are more versatile than ordinary bits (or cbits) because they allow massively parallel computations to be carried out. For example, Shor [1] has demonstrated how a quantum computer operating on qubits can factorize a large number much more efficiently than any classical computer. Fuelled by this and other discoveries, there has been rapid progress in the fields of quantum information theory and quantum computation in recent years. As examples of some of the developments we may mention the construction of logic gates using qubits, [2] the development of quantum error correcting codes [3], and the simulation of a realistic (but limited) quantum computation [4]. Paralleling these theoretical developments, and giving additional impetus to them, are the experimental realizations of qubits in cavity QED [5,2], ion traps [6], optical down-conversion [7], and nuclear spin systems [8].

The purpose of this paper is to step back from these latest developments and ask a simple question about qubits. The question is this: How can Alice convey information about the state of a qubit (which she knows exactly) to Bob over a classical communication channel, using no more than a stipulated number of bits, so that Bob can reproduce the qubit as reliably as possible? By "reliably" we mean that the fidelity of the transmission, F, defined as

$$F = \frac{1}{4\pi}\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, |\langle \hat{n}(\theta,\phi)|\hat{m}\rangle|^2, \tag{1}$$

be as large as possible. In Eq. (1) $|\hat{n}\rangle$ is the state of Alice's qubit and $|\hat{m}\rangle$ is Bob's guess for it, with $\hat{n}$ and $\hat{m}$ being unit vectors along the spin axes of these qubits; the fidelity is Bob's "score" ($= |\langle\hat{n}(\theta,\phi)|\hat{m}\rangle|^2$) averaged over a large number of runs in which the spin axis of Alice's qubit ranges uniformly over the unit sphere (hence the angular integral in Eq. (1)). The question just posed was first raised by Gisin [9] in

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an interesting paper dealing with the relationship between teleportation and quantum non-locality [2]. Gisin answered the question in the special case when Alice could use two bits (but no more) to transmit a message to Bob. Gisin suggested that Alice and Bob adopt the following three-part strategy: (i) They divide the surface of the unit sphere into four equal triangular areas based on a regular tetrahedron; (ii) Alice notes in which of these areas the spin vector of her qubit lies and transmits this information to Bob using the two bits at her disposal; and (iii) Bob chooses as his estimate of the qubit the state at the center of the area indicated to him by Alice. If this procedure is repeated many times, with Alice's qubits distributed uniformly over the unit sphere, Gisin showed that the average fidelity with which Bob could reproduce Alice's qubit is 0.87.

Since the use of two bits already permits a fidelity of 0.87 to be attained, one might suspect that with just a few additional bits a fidelity close to 1 (representing perfect transmission) can be achieved. More generally, Gisin's result suggests the following problems: (i) If Alice can use no more than x bits to transmit a (classical) message to Bob, what is the maximum fidelity with which he can estimate her qubit and what strategy should they follow to ensure this? [3] (ii) If Bob is to estimate Alice's qubit with a specified fidelity $1-\epsilon$, what is the smallest number of bits that Alice must use in sending her message and what strategy should they follow? In posing these problems it is assumed that the qubit to be transmitted (which we refer to as Alice's qubit) is distributed with a known probability on the unit sphere. In this paper we consider only the case in which Alice's qubits are distributed uniformly over the unit sphere, although other distributions could be considered. Both the above problems are variants of optimization problems on the sphere that mathematicians have been tackling for many years. The general problem of this kind is to find a set of points on the sphere for which a certain energy or "cost" function is extremized. Well-studied examples of such problems are the placement of N point charges on a sphere so that their total electrostatic energy is minimized [11] and packing and covering problems on the sphere [12]. An overview of such problems from many different fields can be found in Ref. [13].

Some work in quantum information theory bearing on the subject of this note should also be mentioned. The problem of determining the state of an unknown qubit of which a finite number of copies is available is an interesting (and difficult) one which has been addressed by Peres and Wootters [14], Massar and Popescu [15], and Peres [16]. The use of qubits to encode messages from a quantum signal source was considered in a fundamental paper by Schumacher [17], who proved a quantum version of Shannon's noiseless coding theorem. Recently Schumacher et al. [18] proved a generalization of Kholevo's theorem concerning the amount of accessible information in a quantum channel.

We have not been able to obtain general solutions to either of the two problems posed above. However, we have obtained answers to the first problem for particular values of x based on a simple generalization of Gisin's strategy: We divide the unit sphere into N polygonal regions (with $N = 2^x$ being an integer), require Alice to tell Bob in which region her qubit lies, and require Bob to approximate the qubit by the state at the center of the region indicated to him by Alice. We have used this strategy to calculate the average fidelity (F) and worst case fidelity (F_W) when N equals the number of faces of any regular or semiregular polyhedron by dividing up the unit sphere into regular polygons based on the polyhedron. The average fidelity F, discussed earlier, is just the fidelity averaged over all possible (uniformly distributed) values of Alice's qubit on the unit sphere. The worst case fidelity, F_W, is defined as the lowest fidelity than can occur in a single transmission. The worst case fidelity arises when Alice's qubit lies at a corner of one of the polygonal regions into which the unit sphere is divided, for it is then furthest from Bob's estimate at the center of the polygon. Table 1 shows the fidelities F and F_W for several values of $N = 2^x$ obtained using the above strategy. The Appendix briefly describes the calculations leading to these results.

For completeness, we have begun Table 1 by listing the fidelities when Alice transmits no information or one bit of information to Bob. The next five entries, for $N = 4, 6, 8, 12$ and 20, were obtained by di-

[2] Fundamental questions about teleportation and non-locality are addressed in Ref. [10].

[3] This problem can also be restated as follows: Find N points on a sphere such that if a dart is thrown at random on the sphere, the average of its squared distance from the closest of the points is as small as possible.

Table 1
Fidelities for transmission of qubits based on regular and semi-regular divisions of the sphere. N = number of faces of polyhedron in Column 1, $x = \log_2 N$, F = average fidelity and F_W = worst case fidelity

Polyhedron	N	x	F	F_W
sphere	1	0	0.5000	0
hemisphere	2	1	0.7500	0.5000
tetrahedron 3^3	4	2	0.8724	0.6667
cube 4^3	6	2.5850	0.9156	0.7887
octahedron 3^4	8	3.0000	0.9330	0.7887
dodecahedron 5^3	12	3.5850	0.9579	0.8973
icosahedron 3^5	20	4.3219	0.9716	0.8973
truncated octahedron 4×6^2	14	3.8074	0.9572	0.8873
truncated icosahedron 5×6^2	32	5	0.9837	0.9575
icosidodecahedron $(3 \times 5)^2$	32	5	0.9761	0.9253
great $4 \times 6 \times 10$ rhombicosidodecahedron	62	5.9542	0.9853	0.9525
snub $3^4 \times 5$ dodecahedron	92	6.5236	0.9911	0.9594

viding the sphere into congruent polygons based on the regular polyhedra. For $N = 20$, for example, the sphere is divided into twenty congruent equilateral triangles by projecting the edges of a regular icosahedron onto its circumscribing sphere. In this case Alice uses $\log_2 20 = 4.3219$ bits to convey a message to Bob, who can estimate her qubit with an average fidelity of 0.9716. Although we do not have a rigorous proof, we suspect that for these N values the fidelities we have found are optimal. To illustrate this remark, we compare the icosahedral division of the sphere with another 20 element division based on four equally spaced circles of latitude and two orthogonal circles of longitude. The average fidelity in the latter case is found to be 0.7950, which is considerably worse than for the icosahedral division.

We pass next to strategies based on the semi-regular polyhedra, of which there are thirteen different kinds (excluding prisms and antiprisms). However, only some of these polyhedra have been used in the calculations of Table 1. Consider, for example, the truncated icosahedron (or "buckyball") whose 32 faces consist of 12 pentagons and 20 hexagons. This solid is assigned the symbol 5×6^2 because one pentagon and two hexagons meet at each vertex. A division of the sphere into 32 regions based on this solid allows Alice to send a 5 bit message to Bob as a result of which he can estimate her qubit with an average fidelity of 0.9837. It is interesting to note that there is another solid with 32 faces, the icosidodecahedron, which, however, leads to a slightly smaller average fidelity. The semi-regular solid with the largest number of faces (= 92) is the snub dodecahedron, and it allows an average fidelity of 0.9911 to be achieved (with a worst case fidelity of 0.9594). While we have no proof that these fidelities are optimal, we suspect that for any N for which one or more semi-regular polyhedra exist, the optimal fidelity will be the largest among these values. It is also interesting to note from the last column of Table 1 how the values of F_W keep increasing with increasing N.

It is possible to consider even larger values of N for which higher fidelities F and F_W can be achieved. This may be done, for example, by considering equal area partitionings of the sphere [13] or divisions based on geodesic domes [19]. However, it is far from clear that these methods are optimal.

What is the significance of the above results? In the first place they provide a rather direct connection between cbits and qubits, the fundamental units of classical and quantum information storage, respectively. Since a qubit is represented by a point on the unit sphere, which is a continuous manifold, an infinite number of cbits is strictly needed to capture it precisely. However, the above results indicate how a qubit can be approximated by a finite number of cbits when various levels of error in the result can be tolerated. In particular, we find that a qubit distributed uniformly over the unit sphere can be approximated by a little over $6\frac{1}{2}$ cbits with an average fidelity exceeding 99%. It should be stressed that this equivalence between qubits and cbits does not imply that a qubit can be replaced in a physical process by a certain number of cbits. A qubit is an indispensable physical resource that cannot be simulated by any number of cbits. The above results merely illustrate how information about a known qubit can be transmitted from one place to another by classical means when constraints on the message size and fidelity are in effect.

A second interpretation of these results is connected with the use of teleportation to infer the non-local character of an entangled quantum state. This interpretation was suggested for the $N = 4$ case of this problem by Gisin [9] and goes as follows. In standard teleportation [20], an unknown qubit is transmitted from sender to receiver by a quantum channel supplemented by two bits of classical information. If the quantum channel is an EPR singlet the fidelity of transmission is unity, but if it is any other two-particle state (pure or mixed) the fidelity falls below unity. Gisin argued that if an arbitrary two-particle state is used as the quantum channel, its non-locality can be inferred with certainty only if the fidelity of teleportation exceeds 0.8724 (which is the entry corresponding to $N = 4$ in Table 1). Gisin argued for the sufficiency, but not the necessity, of this condition. The other entries in Table 1 can be given the same interpretation as that of Gisin if the classical message in the teleportation process involves an arbitrary number, x, of bits. However, it should be admitted that this interpretation is somewhat academic because no methods of teleportation have been proposed to date in which the classical message requires more than two bits.

We now consider a slight generalization of the above problem. The complete state of a qubit, including its global phase, can be expressed as $|\Psi(\theta,\phi,\chi)\rangle = e^{i\chi}[\cos(\theta/2)|0\rangle + e^{-i\phi}\sin(\theta/2)|1\rangle]$, where the global phase χ assumes values between 0 and 2π. The global phase of a qubit seems to play no role in quantum computation but it is of interest in other contexts, e.g., in connection with the geometric phase. The problem we wish to address is that of transmitting all three characteristics (θ, ϕ and χ) of a qubit using as few bits and making as small an error as possible. We must first decide on a suitable measure of fidelity to use in gauging the success of a transmission. If θ', ϕ' and χ' are the parameters of the qubit estimated by Bob, we take the fidelity of the transmission to be $\cos^2[\frac{1}{2}(\chi-\chi')]|\langle\Psi'|\Psi\rangle|^2$ where the unprimed (or primed) quantities refer to Alice's original (or Bob's estimated) qubit. This measure of fidelity is clearly not unique but it possesses some very reasonable properties. It becomes equal to unity if, and only if, Bob's estimate coincides with Alice's qubit and it vanishes if the two qubits are orthogonal or differ in their global phases by π. More generally, this measure penalizes deviations in both the global phase and the θ,ϕ parameters of the qubit.

Consider now the problem of transmitting a qubit whose θ,ϕ parameters are distributed uniformly over the unit sphere and whose global phase χ is distributed uniformly over the interval $(0,2\pi)$. This can be done quite straightforwardly. The unit sphere corresponding to the variables θ,ϕ can be divided into equal parts in the manner discussed earlier (for example, a twenty region icosahedral division can be chosen) while the interval from 0 to 2π for χ can be divided into n equal subintervals. To carry out the transmission Alice informs Bob in which of the polygons on the sphere and which of the subintervals of the $(0,2\pi)$ line the qubit lies, and Bob chooses as the parameters of the qubit the center values of the regions indicated to him by Alice. It is easy to see that the average fidelity of transmission can be expressed as the product $F = F_1(\theta,\phi)F_2(\chi)$, where $F_1(\theta,\phi)$ is the average fidelity for the θ,ϕ part of the qubit (which we saw how to calculate earlier) and $F_2(\chi)$ is an additional factor arising from the global phase. This additional factor, which depends only on n, represents the extra diminition in the fidelity resulting from the global phase error. For $n = 5$, for example, $F_2 = 0.9677$.

Another variant of the above problem can be posed if one notices that the parameters θ,ϕ and χ of a qubit, with χ restricted to the interval $(0,\pi)$, are in one-to-one correspondence with the points on the surface of a four-dimensional unit sphere. This becomes clearer if one writes the equation of the sphere as $x^2+y^2+z^2+w^2 = 1$ and parameterizes its cartesian coordinates as $x = \sin\chi\sin\theta\cos\phi$, $y = \sin\chi\sin\theta\sin\phi$, $z = \sin\chi\cos\theta$ and $w = \cos\chi$; then, as ϕ ranges between 0 and 2π and θ and χ each range between 0 and π, the coordinates x, y, z and w sweep out the surface of the 4D unit sphere. The parameters θ,ϕ and χ of a qubit cover the surface of the 4D unit sphere twice, once when $0 \leqslant \chi \leqslant \pi$ and a second time when $\pi \leqslant \chi \leqslant 2\pi$. One can now pose the following problem: If the state (θ,ϕ,χ) of a qubit is distributed with uniform probability over the surface of the 4D unit sphere, what strategy should Alice and Bob adopt so that the former can convey knowledge about the qubit accurately to the latter?

The obvious answer is to divide the surface of the 4D sphere into a number of 3D cells of equal volume that tile its surface without overlap; Alice then notes in which cell her qubit lies and supplies this information to Bob, along with the additional information (requiring one bit) on whether $0 \leqslant \chi \leqslant \pi$ or $\pi \leqslant \chi \leqslant 2\pi$; finally, Bob takes as his estimate of the qubit the state at the center of the cell indicated to him by Alice. The tiling of the 4D sphere by 3D cells can be based, for example, on any of the six four-dimensional regular polytopes. If each 3D cell in any such tiling is replaced by its circumscribing sphere, the surface of the 4-sphere gets covered entirely by overlapping 3-spheres; this "spherical covering" reduces the fidelity (thus yielding a lower bound to it) but it has the advantage of making the fidelity much easier to estimate. As before, one finds that the average fidelity can be written as the product $F = F_1(\theta,\phi)F_2(\chi)$ where the calculation of F_1 was discussed earlier and the extra factor arising from the global phase is now $F_2 = \frac{1}{2} + \frac{2}{3}\sin^3(\chi_0)/[2\chi_0 - \sin(2\chi_0)]$, with χ_0 being the radius of the covering spheres. The value of χ_0 is determined by the regular polytope on which the spherical covering is based. As a concrete example, suppose that Alice and Bob choose a spherical covering of the 4-sphere based on the 120-cell (whose cells are dodecahedra) together with an icosahedral division ($N = 20$) for the θ, ϕ variables. Then the number of bits that Alice would use to send a message to Bob would be $1+\log_2 20+\log_2 120 = 12.2288$. The radius χ_0 would be [21] 0.3881, leading to $F_2 = 0.9777$, and, since $F_1 = 0.9716$, the average fidelity F would be 0.9499.

1. Appendix

We briefly outline the calculation leading to the fidelities reported in Table 1. Any semi-regular polyhedron[4] has two or three kinds of regular polygons for its faces, with an identical arrangement of polygons at each vertex. If we denote by q_i the number of polygons with p_i sides that meet at each vertex, the polyhedron is completely characterized by the numbers (p_1,q_1,p_2,q_2) or $(p_1,q_1,p_2,q_2,p_3,q_3)$ (a slightly more informative notation, used in Table 1, is to note the polygons that are encountered as a vertex is circled in a definite sense). The number of polygons with p_i sides in the polyhedron is given by $f_i = (2q_i/p_i)/(1+R-Q/2)$, where $R = \sum_i q_i/p_i$ and $Q = \sum_i q_i$. A tiling of the unit sphere based on a semi-regular polyhedron leads to the average fidelity

$$F = \frac{1}{4\pi}\sum_i 2p_i f_i \int_0^{\pi/p_i} \mathrm{d}\phi \int_{\mu_i}^{1} \mathrm{d}\mu\, \tfrac{1}{2}(1+\mu), \tag{2}$$

[4] Some useful references on semi-regular polyhedra are in Refs. [22,19].

where $\frac{1}{2}(1+\mu) \equiv \cos^2(\theta/2)$ is the fidelity with which a qubit at angular distance θ from the center of the nearest polygonal face is transmitted. In setting up Eq. (2), the integral of $\cos^2(\theta/2)$ over a polygonal face has been written as $2p_i$ times the integral over an elementary right angled triangle into which the face can be divided. Also, in doing the integrals over any face, $\theta = 0$ is chosen at the center of the face and $\phi = 0$ as the meridian through the center of the face and the midpoint of an edge. The lower limit of the μ ($= \cos\theta$) integration in Eq. (2) is $\mu_i = \cos\alpha_i \cos\phi/(1-\cos^2\alpha_i \sin^2\phi)$, where α_i is the angular separation between the center of the face p_i and the midpoint of any of its edges (this quantity is the same both for the polyhedron and its projection on the sphere). On doing the integrals in Eq. (2) and using some spherical trigonometry to relate the sides and angles of the fundamental right angled triangle, one finds that the expression for the average fidelity can be cast into the form

$$F = \frac{1}{4\pi(1+R-Q/2)}\Big((2R-Q)\pi + 2\sum_i q_i \arcsin[\sec a \cos(\pi/p_i)] + a\tan a \sum_i q_i \cot(\pi/p_i)\Big), \tag{3}$$

where a is half the arc length of an edge of the polyhedron projected onto the unit sphere. For a regular polyhedron, $R = p/q$, $Q = q$ and $\cos a = \cos(\pi/p)/\sin(\pi/q)$, and expression (3) simplifies considerably. For a semi-regular polyhedron, the quantity a still remains to be calculated. Let δ_i be half the vertex angle of the polygonal face p_i of the polyhedron when it is projected onto the unit sphere. The angles δ_i and arc length a can be calculated from the solution of the simultaneous equations

$$\cos a = \frac{\cos(\pi/p_i)}{\sin\delta_i}, \qquad \sum_i q_i\delta_i = \pi. \tag{4}$$

These equations can be solved in closed form for all thirteen semi-regular polyhedra. The values of $\cos(2a)$ (with $0 < 2a < \pi/2$) for the semi-regular solids considered in Table 1 are $4/5$ for 4×6^2, $(80+9\sqrt{5})/109$ for 5×6^2, $(1+\sqrt{5})/4$ for $(3\times 5)^2$, and $(179+24\sqrt{5})/241$ for $4\times6\times10$, while for the snub dodecahedron $(3^4\times5)$ it is given by

$$\cos(2a)=\frac{72-36^{1/3}(\beta_+^{1/3}+\beta_-^{1/3})^2}{144-36^{1/3}(\beta_+^{1/3}+\beta_-^{1/3})^2}, \tag{5}$$

with $\beta_\pm = 9(1+\sqrt{5})\pm\sqrt{102+162\sqrt{5}}$. When these values of a are substituted into Eq. (3), the fidelities listed in Table 1 readily follow. The worst case fidelity for the regular polyhedron (p,q) is $F_W=\frac{1}{2}[1+\cot(\pi/p)\cot(\pi/q)]$ while for a semi-regular polyhedron it is $F_W=\frac{1}{2}[1+\cot(\pi/p_{max})\cot(\delta_{max})]$ where p_{max} is the largest of the p_i in the polyhedron and δ_{max}, which is the semi-vertex angle of the corresponding face, can be calculated from Eq. (4) and the known value of a. It is worth noting that the fidelities F and F_W for any polyhedron follow completely from the numbers (p_i,q_i) characterizing it.

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