

## Decoherence of a measure of entanglement

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For a solvable pure-decoherence model, we confirm by an explicit model calculation that the decay of entanglement of two two-state systems (two qubits) is approximately governed by the product of the suppression factors describing decoherence of the subsystems, provided that they are subject to uncorrelated sources of quantum noise. This demonstrates an important physical property that separated open quantum systems can evolve quantum mechanically on time scales larger than the times for which they remain entangled.

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Entanglement of quantum-mechanical states, referring to the nonlocal quantum correlations between subsystems, is one of the key resources in the field of quantum information science. Many protocols in quantum communication and quantum computation are based on entangled states [1]. When one considers practical applications of entanglement, the coupling of the quantum system and its subsystems to the environment, resulting in decoherence, should be taken into account. It is known [2,3] that entanglement cannot be restored by local operations and classical communications once it has been lost, so understanding of the dynamics of decoherence of entanglement is of importance in many applications.

There are two basic issues in the physics of the loss of entanglement by decoherence, that, while intuitively suggestive, thus far have allowed little quantitative, model-based understanding. To define them, let us refer to two subsystems,  $S^{(1)}$  and  $S^{(2)}$ , of the combined system,  $S$ . The first property of interest is the expectation that when the systems are separated, in that they are subject to independent sources of noise, e.g., when they are spatially far apart, then the decoherence of entanglement is faster [4–6] than the loss of coherence in the quantum-mechanical behavior of each of the subsystems. Thus, the subsystems can for some time still behave approximately in a coherent quantum-mechanical manner, but without correlation with each other.

In order to define the second property of interest, let us point out that the definition of “decoherence” of an open quantum system is not unique. One has to consider the overall time-dependent behavior of the reduced density matrix of the system, obtained for a model of the environmental modes, which are the source of noise and are traced over. This time dependence can involve an oscillatory behavior corresponding to the initial regime of approximately coherent evolution, with frequencies determined by the energy gaps of the system (which can be shifted by the noise). At the

same time, there will be irreversible, decay-type time dependences manifest for larger time scales, which can in many cases be identified with processes such as relaxation, thermalization, pure decoherence, etc., that represent irreversible noise-induced behaviors [7–16].

One, by no means unique, way to quantify the degree of loss of coherence is by the decay of the absolute values of off-diagonal elements of the reduced density matrix. This definition is only meaningful at relatively late stages of the dynamics, when the density matrix has already become nearly diagonal in a basis favored by external and internal interactions, and by environmental influences, e.g., for thermalization, the energy basis. More careful definitions of measures of decoherence are possible [17], but we will use the off-diagonal-element nomenclature for clarity. Recent experimental NMR studies [18] have considered various “orders of coherence” that involve off-diagonal elements, for systems of up to 650 spins.

The second property of interest is formulated in this language as follows. For noninteracting and nonentangled subsystems, the density matrix of the whole system will be a direct product of the subsystem density matrixes. In this simple case, there will be far-off-diagonal density-matrix elements of the system that will decay by a factor that is a product of the decay factors of the subsystem off-diagonal elements. Specifically, if the large time decay is exponential, then the decay *rates* will be additive [19,20].

A related “additivity” property has been mathematically explored for certain measures of initial decoherence [17], for *entangled* subsystems. Recently, exploration of the following physically very suggestive question has been initiated [5]: If we know the suppression factors,  $0 \leq \delta^{(1,2)} \leq 1$ , that roughly measure decoherence for the two subsystems, then are there any *physically meaningful* quantities that are suppressed by the product  $\delta^{(1)}\delta^{(2)}$ ? The other suggestive alternative is that the “worst case scenario” for physically relevant loss-of-coherence measures of the combined system is suppression by the factor of  $\min(\delta^{(1)}, \delta^{(2)})$ . The two alternatives are, of course, only approximate, qualitative statements, possibly for upper bounds for oscillatory quantities, because we have not specified the precise measures to use, or the dependence

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on (or maximization of the decay rate over) the initial conditions.

In this work, we show by an explicit calculation for a solvable pure-decoherence model of two qubits (two-state systems, spins-1/2) interacting with a bath of bosonic modes, that the measure of entanglement introduced in [21], is indeed suppressed by the factor  $\delta^{(1)}\delta^{(2)}$ . We focus on the two-qubit system, because it is only for this simplest case that an explicit expression for a measure of entanglement called concurrence was obtained [21]. Our study expands the recent works [4,5] that considered similar properties for different models. We are able to derive explicitly the product of suppression factors result.

For brevity, from now on we will use subscripts or superscripts  $r=1, 2$  to label the spins (two-level subsystems),  $H_S^r = \mathcal{A}^r \sigma_z^r$ . Each spin interacts with a bosonic bath of modes  $H_B^r = \sum_k \omega_k^r b_k^{r\dagger} b_k^r$ , which has been widely used [8,12,14] as a model of quantum noise (we set  $\hbar=1$ ). The interaction between the quantum systems and the environment is taken in the form  $H_I^r = \sigma_z^r \sum_k (g_k^{r*} b_k^r + g_k^r b_k^{r\dagger})$ . This choice, corresponding to  $[H_B^r, H_I^r]=0$ , leads to a solvable model and has been identified as an appropriate description of pure decoherence [14].

We assume that there is no interaction between the qubits, so that the Hamiltonian of the whole system has the form  $H = \sum_r (H_S^r + H_B^r + H_I^r)$ . The main reason for this assumption is, of course, to have a solvable model. In addition, we point out that qubit-qubit interactions, either direct or those induced by the bath modes, can decrease or *increase* their entanglement. For the latter reason, we also assumed that the noise is uncorrelated at the two-qubit locations, namely the bath modes are independent for each qubit (the most natural situation is when the qubits are spatially separated).

The initial state of the two qubits, described by the density matrix  $\rho_S(0)$ , can be entangled. However, we assume [12,14] that the qubits are initially not entangled with the bath modes. The overall initial density matrix is then

$$\rho(0) = \rho_S(0) \otimes \rho_B^1(0) \otimes \rho_B^2(0). \quad (1)$$

The reservoirs are in thermal equilibrium at the temperature  $T$  (with  $\beta \equiv 1/k_B T$ ),

$$\rho_B^r(0) = \prod_k (1 - e^{-\beta\omega_k^r}) e^{-\beta\omega_k^r b_k^{r\dagger} b_k^r}. \quad (2)$$

The total Hamiltonian is time independent, so the reduced density matrix of the two qubits at time  $t \geq 0$  is

$$\rho_S(t) = \text{Tr}_B[U\rho(0)U^\dagger], \quad (3)$$

where the evolution operator factorizes,  $U = e^{-iHt} = U_1 U_2$ . The trace over the bosonic modes of the two baths,  $\text{Tr}_B$  in (3), can then be evaluated exactly by using the techniques of [7,16].

It is convenient to write the density operator  $\rho_S(t)$  in the matrix form,

$$\rho_S^{\gamma_1^r \gamma_1^s, \gamma_2^r \gamma_2^s}(t) \equiv \langle \gamma_1^r \gamma_1^s | \rho_S(t) | \gamma_2^r \gamma_2^s \rangle, \quad (4)$$

where  $\gamma_q^r = \pm 1$  has two indexes:  $r$  labels the qubit, while  $q$  simply indicates whether it marks row or column matrix element positions. The values  $+1$  and  $-1$  correspond to the spin states  $\uparrow$  and  $\downarrow$ , respectively.

After several straightforward transformations, (3) is reduced to

$$\rho_S^{\gamma_1^r \gamma_1^s, \gamma_2^r \gamma_2^s}(t) = e^{iA^1(\gamma_2^1 - \gamma_1^1)t + iA^2(\gamma_2^2 - \gamma_1^2)t} T^{\gamma_1^1 \gamma_1^2} T^{\gamma_2^1 \gamma_2^2} \rho_S^{\gamma_1^r \gamma_1^s, \gamma_2^r \gamma_2^s}(0), \quad (5)$$

where the coefficients are

$$T^{\gamma_1^r \gamma_2^r} = \text{Tr}_B[e^{-i(H_B^r + \gamma_1^r \tilde{H}_I^r)t} \rho_B^r e^{i(H_B^r + \gamma_2^r \tilde{H}_I^r)t}]; \quad (6)$$

here  $\tilde{H}_I^r$  is defined by  $H_I^r = \sigma_z^r \tilde{H}_I^r$ . Utilizing the identities from [7,16], we find an explicit expression,

$$T^{\gamma_1^r \gamma_2^r} = \exp[-G_r(t)(\gamma_1^r - \gamma_2^r)^2], \quad (7)$$

where  $G_r(t)$  is the well-studied spectral function [8,13],

$$G_r(t) = 2 \sum_k \frac{|g_k^r|^2}{(\omega_k^r)^2} \sin^2 \frac{\omega_k^r t}{2} \coth \frac{\beta \omega_k^r}{2}. \quad (8)$$

A general property of the pure-decoherence models [14] is that the diagonal elements of the density matrix will stay unchanged during the evolution.

Utilizing the new variables  $p_r \equiv e^{2iA^r t}$  and  $q_r \equiv e^{-4G_r(t)}$  the density matrix can be written explicitly,

$$\rho_S(t) = \begin{pmatrix} \rho_S^{\uparrow\uparrow, \uparrow\uparrow}(0) & p_2 q_2 \rho_S^{\uparrow\uparrow, \uparrow\downarrow}(0) & p_1 q_1 \rho_S^{\uparrow\uparrow, \downarrow\downarrow}(0) & p_1^* q_1 p_2^* q_2 \rho_S^{\uparrow\uparrow, \downarrow\downarrow}(0) \\ p_2 q_2 \rho_S^{\uparrow\downarrow, \uparrow\uparrow}(0) & \rho_S^{\uparrow\downarrow, \uparrow\downarrow}(0) & p_1^* q_1 p_2 q_2 \rho_S^{\uparrow\downarrow, \downarrow\downarrow}(0) & p_1^* q_1 \rho_S^{\uparrow\downarrow, \downarrow\downarrow}(0) \\ p_1 q_1 \rho_S^{\downarrow\uparrow, \uparrow\uparrow}(0) & p_1 q_1 p_2^* q_2 \rho_S^{\downarrow\uparrow, \uparrow\downarrow}(0) & \rho_S^{\downarrow\uparrow, \downarrow\downarrow}(0) & p_2^* q_2 \rho_S^{\downarrow\uparrow, \downarrow\downarrow}(0) \\ p_1 q_1 p_2 q_2 \rho_S^{\downarrow\downarrow, \uparrow\uparrow}(0) & p_1 q_1 \rho_S^{\downarrow\downarrow, \uparrow\downarrow}(0) & p_2 q_2 \rho_S^{\downarrow\downarrow, \downarrow\downarrow}(0) & \rho_S^{\downarrow\downarrow, \downarrow\downarrow}(0) \end{pmatrix}. \quad (9)$$

To analyze the effect of decoherence on the entangled qubit states we use a measure of entanglement. The entanglement of formation [22] was historically the first widely accepted measure of entanglement. For a mixed state  $\rho_S$ , the

evaluation of this measure is related to minimization over all the possible pure-state decompositions of  $\rho_S$ , and even for a two-qubit system getting analytical results for this measure is a complicated problem. The *concurrence* [21] is a quantity

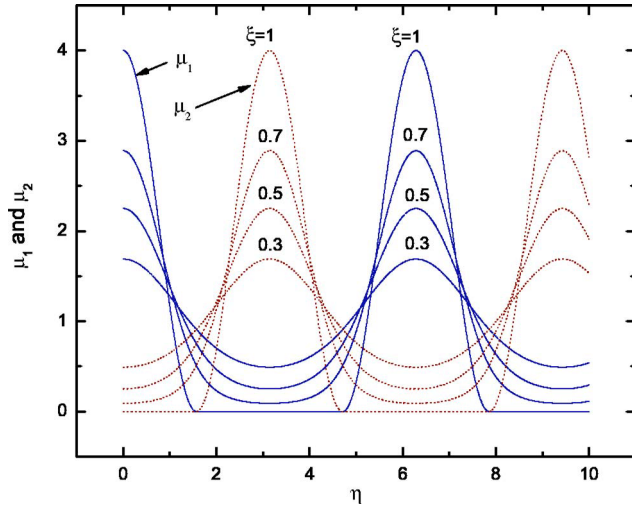


FIG. 1. (Color online) Eigenvalues  $\mu_1$  and  $\mu_2$  as functions of  $\eta$ , for several values of  $\xi$ , with the prefactor  $|\alpha|^2/(1+|\alpha|^2)^2$  suppressed.

monotonically related to the entanglement of formation, hence it may be used as a convenient substitute for it. Given a pure or mixed state,  $\rho_S$ , of two qubits, we define the spin-flipped state

$$\tilde{\rho}_S = (\sigma_y \otimes \sigma_y) \rho_S^* (\sigma_y \otimes \sigma_y), \quad (10)$$

and the Hermitian matrix  $R(\rho_S) = \sqrt{\rho_S \tilde{\rho}_S} \rho_S$  with eigenvalues  $\lambda_{i=1,2,3,4}$ . Then the concurrence [21] is given by

$$C(\rho_S(t)) = \max \left\{ 0, 2 \max_i \lambda_i - \sum_{j=1}^4 \lambda_j \right\}. \quad (11)$$

Since we know  $\rho_S(t)$  explicitly (9), the evaluation of (11) is reduced to finding the eigenvalues of a  $4 \times 4$  matrix.

For illustration, we considered the system of two qubits in a pure state which at time  $t=0$  is  $|\psi\rangle = (|\uparrow\downarrow\rangle + \alpha|\downarrow\uparrow\rangle)/\sqrt{1+|\alpha|^2}$ . Here the (complex) parameter  $\alpha$  characterizes the degree of entanglement. Under the influence of the quantum noise the system evolves from the state  $\rho_S(0) = |\psi\rangle\langle\psi|$  to the mixed state

$$\rho_S(t) = \frac{1}{1+|\alpha|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & p_1^* q_1 p_2 q_2 \alpha^* & 0 \\ 0 & p_1 q_1 p_2^* q_2 \alpha & |\alpha|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

To evaluate the measure of entanglement at times  $t > 0$ , we have to find the eigenvalues of the matrix  $R$ , which can be obtained from the eigenvalues of the product  $\rho_S(t) \tilde{\rho}_S(t)$ . The latter eigenvalues are

$$\mu_{1,2} = \frac{|\alpha|^2}{(1+|\alpha|^2)^2} (1 + \xi^2 \cos 2\eta \pm 2\xi \cos \eta \sqrt{1 - \xi^2 \sin^2 \eta}), \quad (13)$$

and  $\mu_{3,4} = 0$ . Here  $\xi \equiv e^{-4[G_1(t)+G_2(t)]}$  and  $\eta \equiv 2(\mathcal{A}^2 - \mathcal{A}^1)t$ . Then the eigenvalues  $\lambda_i = \sqrt{\mu_i}$ , and as a result the concurrence takes the form,

$$C(\rho_S(t)) = |\sqrt{\mu_1} - \sqrt{\mu_2}|. \quad (14)$$

The eigenvalues  $\mu_{1,2}$  are shown in Fig. 1. For example, for a simple case of identical qubits,  $\mathcal{A}^2 = \mathcal{A}^1$ , we have  $\eta = 0$  and  $\lambda_{1,2} = |\alpha|(\xi \pm 1)/(1+|\alpha|^2)$ . As a result the concurrence is

$$C_{\eta=0} = \frac{2|\alpha|}{1+|\alpha|^2} e^{-4[G_1(t)+G_2(t)]}. \quad (15)$$

This establishes the product of the suppression factors property alluded to in the introduction, because it is known [14] that each of the exponential factors  $e^{-4G_{1,2}(t)}$  measures the decay of the off-diagonal matrix elements when each qubit is isolated from the other, but exposed to its own bath. When the qubits are not identical, one can prove that for any  $t \geq 0$ ,

$$C_\eta(t) \leq C_{\eta=0}(t), \quad (16)$$

so that the product of the factors property applies as an upper bound. The recent Markovian-approximation results [5,6], appropriate for large times, have yielded an interesting observation that for some initial conditions the concurrence, unlike coherence, can drop to zero in finite time [23–25]. We have not explored this property within the pure-decoherence scheme considered here.

In summary, we connected two important issues in the studies of entanglement and decoherence; namely, for a solvable pure-decoherence model, we confirmed that the decay of entanglement is approximately governed by the product of the suppression factors describing decoherence of the subsystems, provided that they are subject to uncorrelated sources of noise. Our results also suggest avenues for future work. Specifically, for multiqubit systems, one might speculate that similar arguments could apply “by induction.” However, understanding of entanglement is far from intuitive, especially when one considers more than two two-state systems. Therefore, for any definitive progress, one has first to develop appropriate quantitative measures of entanglement, and then try to quantify entanglement and decoherence in a unified way.

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