CONVERGENCE ANALYSIS FOR THE NUMERICAL BOUNDARY CORRECTOR FOR ELLIPTIC EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS

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Abstract. We develop the convergence analysis of a numerical scheme for approximating the solution of the elliptic problem

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i} a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega,$$

where $a(y) = (a_{ij}(y))$ is a periodic symmetric positive definite matrix and $\Omega = (0,1)^2$. The major goal of the numerical scheme is to capture the ϵ-scale of the oscillations of the solution u_ϵ on a mesh size $h > \epsilon$ (or $h >> \epsilon$). The numerical scheme is based on asymptotic expansions, constructive boundary corrector and finite element approximations. New a priori error estimates are established for the asymptotic expansions and for the constructive boundary correctors under weak assumptions on the regularity of the problem. These estimates permit to establish sharp finite element error estimates and to consider composite materials applications. Depending on the regularity of the problem, we establish for the numerical scheme a priori error estimates of $O(h^2 + \epsilon^{3/2} + \epsilon h)$ on the L^2 norm, and $O(h + \epsilon^{1+\hat{\delta}})$ for the broken H^1-norm where $\hat{\delta} \in (-\frac{1}{4}, 0]$.

Key words. Finite elements, homogenization, elliptic equations, multiscaling, boundary layer, mixed finite elements, discontinuous coefficients, composite materials.

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1. Introduction.

This paper develops the convergence analysis of the numerical scheme proposed in [46] to approximate the solution of the following problem:

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i}(a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon) = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here, ϵ is a small scale and $a(y) = (a_{ij}(y))$ is a periodic symmetric positive definite matrix with period $Y = (0,1)^2$, and $\Omega = (0,1)^2$. We assume that $a_{ij} \in L^\infty_{\text{per}}(Y)$, i.e., a_{ij} is Y-periodic and $a_{ij} \in L^\infty(\mathbb{R}^2)$, and that there exists a positive constant γ_a such that $\gamma_a \|\xi\|^2 \le a_{ij}(y)\xi_i\xi_j$ for all $\xi \in \mathbb{R}^2$ and $y \in Y$. Throughout the text we consider the the Einstein summation convention, i.e., repeated indices indicate summation, and domains are always considered open.

We note that standard finite element methods do not yield good numerical approximations when the mesh size $h > \epsilon$; see [27]. To overcome this, recently new numerical methods have been proposed for solving the problem (1.1) such as the multi-scale finite element methods [22, 26, 4, 15, 23], the residual-free bubble function methods [13, 6, 5, 39, 14], and the generalized FEM for homogenization problems [40]. There are also related methods for the case in which the homogenized equation is not known; see the heterogeneous multiscale method [18, 19, 1] and [20, 21]. The numerical method considered here, opposed to the methods in [6, 26, 39, 4, 13], is based strongly on the asymptotic expansion of u_ϵ. We also explore the periodicity of the matrix a to obtain a very efficient numerical method for approximating u_ϵ.

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A theoretical tool commonly used to treat this problem is homogenization theory [8, 9, 10, 11, 16, 28, 37] where for instance, a first order term expansion for u_ϵ plus a boundary corrector term are considered. Based on such theory, [45, 46] propose numerical schemes to approximate such terms. These numerical methods were designed to work with a mesh size $h > \epsilon$ (or $h >> \epsilon$), however, they also work for the case $h < \epsilon$. In [45] is presented a numerical algorithm for the case the domain Ω is a rectangular region, while [46] generalizes the method to the case where the domain Ω is a convex polygon with rational boundary normals.

The convergence analysis for the numerical method is performed in two parts. In the first part we estimate the error between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$ in L^2 and H^1 norms, where ϕ_ϵ is a constructive approximation for the boundary corrector term θ_ϵ; see [3, 35, 36]. To estimate this error, we first estimate the error between u_ϵ and $u_0+\epsilon u_1+\epsilon\theta_\epsilon$ on the H^1 and L^2 norms, see Propositions 3.1 and 3.4, and then estimate the the error between ϕ_ϵ and θ_ϵ; see Propositions 3.2 and 3.3. The Propositions 3.1 and 3.4 extend some results of [3, 35] using less regularity requirements. More specifically, the Proposition 3.1 gives the same error estimate of Theorem 2.2 [3], however, here we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \leq 1/2$ and for Lipschitz domains $\Omega \subset \mathbb{R}^{2,3}$, while in Theorem 2.2 [3] it is assumed that $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in H^1_{per}(Y)$, for Lipschitz domains $\Omega \subset \mathbb{R}^d$, d a natural number. We also note that the Propositions 3.1 and 3.4 generalize respectively, the Propositions 2.1 and 2.3 of [35]. In the Proposition 3.1 we assume that $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \leq 1/2$, and $\Omega \subset \mathbb{R}^{2,3}$ a Lipschitz domain, while in the Proposition 2.1 [35] it is assumed that $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$ a smooth domain. The Proposition 3.4 assumes that $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, χ^j and $\chi^{ij} \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \leq 1/2$, $\Omega \subset \mathbb{R}^{2,3}$ a Lipschitz domain, while the Proposition 2.3 [35] assumes that $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^3(\Omega)$ and $\Omega \subset \mathbb{R}^2$ smooth.

The importance of considering a theory that handles the case $a_{ij} \in L^\infty_{per}(Y)$ comes from applications to composite materials where the coefficients a_{ij} are often piecewise constants; see also the Theorem 1.1 [31] which gives conditions on the discontinuities of the functions a_{ij} so that χ^j and $\chi^{ij} \in W^{1,\infty}_{per}(Y)$. We also observe that the Proposition 2.1 [35] is used in the convergence analysis of the numerical methods presented in [22, 27, 39], and therefore, the analysis presented here can be used to extend the analysis of these numerical methods with less regularity requirements on a, u_0 or Ω. To the best of our knowledge, the Propositions 3.2 and 3.3 have not been considered in the literature, however a technique developed in [35] is used in part of the proof of the Proposition 3.2.

In the second part of this paper we develop the finite element analysis to estimate the a priori error estimates due to the finite element approximations. One difficulty lies in the fact that we need a discrete approximation for $\partial_\eta u_0$ to define the Dirichlet boundary condition for the discrete boundary corrector problem; if u_0^h is a finite element approximation for u_0, then $\partial_\eta u_0^h$ does not necessary belong to $H^{1/2}(\partial\Omega)$, i.e., the space required to define the boundary corrector terms. To overcome this, we introduce a Lagrange multiplier space to approximate $\partial_\eta u_0$ and to develop error estimates between $\partial_\eta u_0$ and its discrete approximation μ_h using fractional-order Sobolev norms. The Lagrange multiplier plays an important role to obtain convergence rates under weak assumptions on the regularity of u_0 and χ^j.

We now introduce some norms and semi-norms. Let $B \subset \mathbb{R}^{2,3}$ be a Lipschitz domain and define

$$\|v\|_{m,\infty,B} = \max_{|\alpha|\leq m}\{\text{ess.}\sup_{x\in B}|\partial^\alpha v(x)|\} \quad\text{and}\quad |v|_{m,\infty,B} = \max_{|\alpha|=m}\{\text{ess.}\sup_{x\in B}|\partial^\alpha v(x)|\},$$

and for $1 \leq q < \infty$

$$\|v\|_{m,q,B} = \left(\int_B \sum_{|\alpha|\leq m}|D^\alpha v|^q dx\right)^{1/q} \quad\text{and}\quad |v|_{m,q,B} = \left(\int_B \sum_{|\alpha|=m}|D^\alpha v|^q dx\right)^{1/q}.$$

We also define the non-conforming norms related to a partition $\mathcal{T}_h = K_1, K_2, ..., K_N$ of B by

$$\|v\|_{m,h} = \sqrt{\sum_{K_j\in\mathcal{T}_h}\|v\|^2{}_{H^m(K_j)}}.$$

Throughout this paper when we do not make reference to the domain B or to the norm q, is because $B = \Omega$ or $q = 2$, respectively. In what follows c denotes a generic constant independent of ϵ and mesh parameters.

This paper is organized as follows: we devote Section 2 to introduce the asymptotic expansion of u_ϵ and to describe the continuous approximation for the boundary corrector terms; in Section 3 we develop the main results and proofs associated to error estimates due to the asymptotic expansion approximation. In Section 4 we consider the numerical algorithm introduced in [46] for the case $\Omega = (0,1)^2$, while in Section 5 we analyze the discretization errors due to the finite element approximation. In Section 6 we test numerically the numerical schemes, and in Section 7 we make some conclusions.

2. Periodic Homogenization and Asymptotic Expansions.

2.1. Asymptotic Expansions.

Consider the following anzats

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots, \tag{2.1}$$

where the functions $u_j(x, y)$ are Y periodic in y. Using (2.1) in the equation (1.1) and matching the terms with the same order in ϵ, one may define functions u_j such that $u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon)$ approximates u_ϵ as

$$\|u_\epsilon(x) - u_0(x, x/\epsilon) - \epsilon u_1(x, x/\epsilon)\|_1 \leq c\epsilon^{1/2}\|u_0\|_{2,\infty},$$

where $u_0 \in C^2(\Omega)$ and $\chi^j \in W^{1,\infty}(Y)$, and the constant c depends only on a, χ^j and $\Omega \subset \mathbb{R}^{2,3}$ a Lipschitz domain. These terms are defined in detail below; for more details, including the proof of the above inequality, see [11, 28].

Let $\chi^j \in H^1_{\text{per}}(Y)$, i.e., $\chi^j \in H^1_{loc}(\mathbb{R}^{2,3})$ and Y-periodic, be the weak solution with zero average over Y of

$$\nabla_y \cdot a(y)\nabla_y\chi^j = \nabla_y \cdot a(y)\nabla_y y_j = \frac{\partial}{\partial y_i}a_{ij}(y), \tag{2.2}$$

and define the matrix

$$A_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i)\frac{\partial}{\partial y_m}(y_j - \chi^j)dy. \tag{2.3}$$

It is easy to check that the matrix A is symmetric positive definite. Define $u_0 \in H_0^1(\Omega)$ as the weak solution of

$$-\nabla . A \nabla u_0 = f \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial\Omega, \tag{2.4}$$

and let

$$u_1(x, \frac{x}{\epsilon}) = -\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x). \tag{2.5}$$

Note that $u_0 + \epsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$. To overcome this, the boundary corrector term $\theta_\epsilon \in H^1(\Omega)$ is introduced and defined as the solution of

$$-\nabla \cdot a(x/\epsilon) \nabla \theta_\epsilon = 0 \text{ in } \Omega, \quad \theta_\epsilon = -u_1(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega. \tag{2.6}$$

Hence, we have that $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon \in H_0^1(\Omega)$. The Propositions 3.1 and 3.4 provide error estimates between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon$ in the norms $\|\cdot\|_1$ and $\|\cdot\|_0$, respectively.

We also define the term u_2, see (2.8), which is used in the proof of the Proposition 3.4. Set

$$b_{ij} = -a_{ij} + a_{im} \frac{\partial \chi^j}{\partial y_m} + \frac{\partial}{\partial y_m}(a_{mi}\chi^j)$$

and observe that $\overline{b}_{ij} = A_{ij}$, where $\overline{b}_{ij} = \int_Y b_{ij} dy$. Define $\chi^{ij} \in H_{per}^1(Y)$ as the weak solution with zero average over Y of

$$\nabla_y \cdot a \nabla_y \chi^{ij} = b_{ij} - \overline{b}_{ij} \tag{2.7}$$

and let

$$u_2(x, \frac{x}{\epsilon}) = -\chi^{ij} \left(\frac{x}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_i \partial x_j}(x). \tag{2.8}$$

Note that $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon + \epsilon^2 u_2$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$ and to overcome this the boundary corrector term $\varphi_\epsilon \in H^1(\Omega)$ is introduced as the weak solution of

$$-\nabla \cdot a(x/\epsilon) \nabla \varphi_\epsilon = 0 \text{ in } \Omega, \text{ and } \varphi_\epsilon(x) = -u_2(x, x/\epsilon) \text{ on } \partial\Omega. \tag{2.9}$$

2.2. The Boundary Corrector Terms. When dealing with constructive boundary corrector terms, we assume that $\Omega = (0,1)^2$, although the same theory holds for the case $\Omega = \prod_{i=1}^2 (a_i, b_i)$, $a_i < b_i \in \mathbb{R}$ or convex polygon or polyhedrom domains with rational boundary normals, see [46, 44].

The coefficients $a_{ij}(x/\epsilon)$ and the Dirichlet boundary data $-u_1(x, \frac{x}{\epsilon})$ in the equation (2.6) are highly oscillatory, thus it is not a trivial problem to obtain a good discrete approximation for θ_ϵ. We consider an analytical approximation for θ_ϵ, denoted by ϕ_ϵ, which satisfies the oscillating boundary condition and is suitable for numerical approximation; see [3, 35]. Analytical and numerical approximations for φ_ϵ will not be considered here in this paper.

Denote η as the unity outward normal vector to $\partial\Omega$ and let $\partial_\eta u_0$ be the unity outward normal derivative of u_0 on $\partial\Omega$. In order to define the approximation ϕ_ϵ for θ_ϵ, we first introduce a decomposition for $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$ as

$$-\nabla \cdot a(x/\epsilon)\nabla\tilde{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \tilde{\theta}_\epsilon = (\chi^j(\frac{x}{\epsilon})\eta_j - \chi^*)\partial_\eta u_0 \text{ on } \partial\Omega \tag{2.10}$$

and

$$-\nabla \cdot a(x/\epsilon)\nabla\bar{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \bar{\theta}_\epsilon = \chi^*\partial_\eta u_0 \text{ on } \partial\Omega, \tag{2.11}$$

where $\chi^*|_{\Gamma_k} = \chi^*_k$, $k \in \{e, w, n, s\}$ are properly chosen constants defined in Subsection 2.2.1, and $\Gamma_e = \{1\} \times [0,1]$, $\Gamma_w = \{0\} \times [0,1]$, $\Gamma_n = [0,1] \times \{1\}$, and $\Gamma_s = [0,1] \times \{0\}$ are the edges of the domain $\Omega = (0,1)^2$. In Remark 2.1 we show that $\chi^*\partial_\eta u_0$ and $\chi^j(\frac{x}{\epsilon})\eta_j\partial_\eta u_0$ belong to $H^{1/2}(\partial\Omega)$, therefore, the problems (2.10) and (2.11) are well posed. Later in this section we define the functions $\tilde{\phi}_\epsilon$ and $\bar{\phi}_\epsilon$, which are the approximations for $\tilde{\theta}_\epsilon$ and $\bar{\theta}_\epsilon$ respectively, and define $\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$.

Remark 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and assume $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. We have by Theorem A.2 [38] that $\partial_\eta u_0|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$ and $\|\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \le c\|u_0\|_2$, therefore,*

$$\|\chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \le c \max_k |\chi^*_k| \|u_0\|_2,$$

where $|\chi^*_k|$ can be bounded by $c \max_j \|\chi^j\|_{1,2,Y}$; see the proof of Lemma 4.4 [3]. Note also that $u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$ and $\frac{\partial u_1}{\partial x_l} = -\left(\frac{\partial \chi^j}{\partial x_l}\right)\frac{\partial u_0}{\partial x_j} - \chi^j\left(\frac{\partial^2 u_0}{\partial x_l \partial x_j}\right)$. If we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$ for $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$, $2 \le p, q \le \infty$, then $u_1 \in H^1(\Omega)$. In addition, using that $u_0 \in H^1_0(\Omega)$ and density arguments we also have $u_1|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$.

2.2.1. Calculating the Constants χ^*_k. We now define properly the constants χ^*_k in order to make the function $\tilde{\phi}_\epsilon$ to decay exponentially to zero away from the boundary $\partial\Omega$ and to satisfy the Dirichlet boundary condition $\tilde{\phi}_\epsilon(x) = -u_1(x, \frac{x}{\epsilon}) - \chi^*\partial_\eta u_0(x)$ for $x \in \partial\Omega$.

For $k \in \{e, w, n, s\}$ let $G_k = I^1_k \times I^2_k$, where $I^1_e = I^2_n = (-\infty, 0)$, $I^2_e = I^2_w = (0,1)$, $I^1_w = I^2_s = (0, \infty)$ and $I^1_n = I^1_s = (0,1)$. Associated to each side Γ_k of Ω define the function v_k as the solution of

$$\begin{aligned} &-\nabla_y \cdot a(y_1, y_2)\nabla_y v_k = 0 \text{ in } G_k, \\ &\text{and } \partial_{y_i} v_n \exp(\gamma(-1)^{m_k} y_{n_k}) \in L^2(G_k) \quad i = 1, 2, \end{aligned}$$

where $n_k = 1$ if $k \in \{e, w\}$, $n_k = 2$ if $k \in \{s, n\}$, $m_k = 1$ if $k \in \{e, n\}$, and $m_k = 2$ if $k \in \{w, s\}$. For each $k \in \{e, w, s, n\}$ we impose the following boundary conditions

$$\begin{aligned} &v_e(0, y_2) = \chi^1(1/\epsilon, y_2) \text{ for } 0 < y_2 < 1, && v_w(0, y_2) = -\chi^1(0, y_2) \text{ for } 0 < y_2 < 1, \\ &v_e(y_1, \cdot) \ [0,1]\text{-periodic for } -\infty < y_1 < 0, && v_w(y_1, \cdot) \ [0,1]\text{-periodic for } 0 < y_1 < \infty, \end{aligned}$$

$$\begin{aligned} &v_n(y_1, 0) = \chi^2(y_1, 1/\epsilon) \text{ for } 0 < y_1 < 1, && v_s(y_1, 0) = -\chi^2(y_1, 0) \text{ for } 0 < y_1 < 1, \\ &v_n(\cdot, y_2) \ [0,1]\text{-periodic for } -\infty < y_2 < 0, && v_s(\cdot, y_2) \ [0,1]\text{-periodic for } 0 < y_2 < \infty. \end{aligned}$$

These boundary layer problems have been studied by several authors, see [30, 32, 28, 35]. Theorem 10.1 in Section 10.4 [32] guarantees existence and uniqueness of the v_k and χ^*_k.

2.2.2. Approximating $\tilde{\theta}_\epsilon$. We note by Remark 2.1 that $(u_1(x,\frac{x}{\epsilon})-\chi^*\partial_\eta u_0)|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$. Thus, we can split $\tilde{\theta}_\epsilon = \sum_{k\in\{e,w,n,s\}} \tilde{\theta}^k_\epsilon$ where

$$L_\epsilon \tilde{\theta}^k_\epsilon = 0 \text{ in } \Omega, \quad \text{and} \quad \tilde{\theta}^k_\epsilon = \begin{cases} -u_1(x,\frac{x}{\epsilon}) - \chi^*_k \partial_\eta u_0 & \text{on } \Gamma_k \\ 0 & \text{on } \partial\Omega \setminus \Gamma_k. \end{cases} \tag{2.12}$$

We propose to approximate $\tilde{\theta}^k_\epsilon$ by

$$\tilde{\phi}^k_\epsilon(x_1,x_2) = \varphi_k(x_1,x_2)\left(v_k(\frac{x-\delta_k}{\epsilon}) - \chi^*_k\right)\frac{\partial u_0}{\partial x_{i_k}}(x_1,x_2), \tag{2.13}$$

where $\delta_e = (1,0)$, $\delta_n = (0,1)$, $\delta_w = (0,0)$ and $\delta_s = (0,0)$. The cutting-off functions φ_k are nonnegative smooth functions satisfying $\varphi_k(x_1,x_2) = \varphi_k(x_1)$ for $k \in \{e,w\}$, $\varphi_k(x_1,x_2) = \varphi_k(x_2)$ for $k \in \{n,s\}$, and

$$\varphi_e(s) = \varphi_n(s) = \begin{cases} 1 & \text{if } s \in [2/3,1] \\ 0 & \text{if } s \in [0,1/3], \end{cases} \qquad \varphi_w(s) = \varphi_s(s) = \begin{cases} 0 & \text{if } s \in [2/3,1] \\ -1 & \text{if } s \in [0,1/3]. \end{cases}$$

Therefore,

$$\tilde{\phi}_\epsilon = \sum_{k\in\{e,w,n,s\}} \tilde{\phi}^k_\epsilon \tag{2.14}$$

approximates $\tilde{\theta}_\epsilon$ with $\tilde{\phi}_\epsilon = \tilde{\theta}_\epsilon$ on the boundary of Ω.

2.2.3. Approximating $\bar{\theta}_\epsilon$. The boundary condition imposed on the equation (2.11) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the weak solution of

$$-\nabla \cdot A\nabla\bar{\phi} = 0 \text{ in } \Omega, \qquad \bar{\phi} = \chi^*\partial_\eta u_0 \text{ on } \partial\Omega, \tag{2.15}$$

where Remark 2.1 guaranties the well posedness of this problem.

2.2.4. Approximating u_ϵ. We finally define the analytical approximation for u_ϵ as $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$, where

$$\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}. \tag{2.16}$$

Note that $\phi_\epsilon|_{\partial\Omega} = \theta_\epsilon|_{\partial\Omega}$, therefore, $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon = 0$ on $\partial\Omega$.

3. Convergence Results of Asymptotic Expansions. We now prove the propositions that are used in the proof of the Theorems 3.1 and 3.2.

The following Proposition 3.1 generalizes the Theorem 2.2 [3]. Here we assume that $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \le 1/2$, while the Theorem 2.2 [3] assumes that $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in H^1_{per}(\Omega)$. The Proposition 3.1 also generalizes the Proposition 2.1 [35], where it is assumed that $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$ is a smooth domain. We also refer the Theorem 1.1 [31] for results associated to sufficient conditions on a_{ij} in order to have $\chi^j \in W^{1,\infty}_{per}(Y)$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain with Lipschitz boundary. Let u_ϵ be the solution of the problem (1.1) and u_0, u_1 and θ_ϵ be defined by the equations (2.4), (2.5) and (2.6), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$ for $1/p + 1/q \leq 1/2$, $2 \leq p, q \leq \infty$. Then there exists a constant c, independent of u_0, χ^j and ϵ, such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_1 \leq c\epsilon \max_j \|\chi^j\|_{1,q,Y}\|u_0\|_{2,p}. \tag{3.1}$$

In addition, if $p, q > d$ then

$$\|u_\epsilon - u_0\|_0 \leq c\epsilon \max_j \|\chi^j\|_{1,q,Y}\|u_0\|_{2,p}. \tag{3.2}$$

Proof. Define

$$v_0(x, y) = a(y)\nabla_x u_0(x) + a(y)\nabla_y u_1(x, y) = a(y)(\nabla_y y_j - \nabla_y \chi^j(y))\frac{\partial u_0}{\partial x_j}(x). \tag{3.3}$$

From the definition of χ^j, see (2.2), we have

$$\int_Y \left(a(y)(e_j - \nabla_y \chi^j(y)) - Ae_j\right) \nabla_y \phi(y) dy = 0, \quad \forall\, \phi \in H^1_{per}(Y).$$

Since the vector $a(y)(e_j - \nabla_y \chi^j(y)) - Ae_j$ is Y periodic and has zero average entries over Y, then the Lemma 3.1 guaranties the existence of a $\phi_j(y) \in H^1_{per}(Y)$ with zero average over Y and such that

$$a(y)(\nabla_y y_j - \nabla_y \chi^j(y)) - Ae_j = -curl_y \phi_j(y). \tag{3.4}$$

Let

$$\phi(x, y) = \phi_j(y)\frac{\partial u_0}{\partial x_j}(x). \tag{3.5}$$

For the case $d = 2$ define

$$\begin{aligned} v_1(x, y) &= -curl_x \phi(x, y) \\ &= \begin{pmatrix} -\phi_j(y)\frac{\partial^2 u_0}{\partial x_2 \partial x_j}(x) \\ \phi_j(y)\frac{\partial^2 u_0}{\partial x_1 \partial x_j}(x) \end{pmatrix}, \end{aligned}$$

and note that $|curl_y \phi_j|_{0,q} = |\phi_j|_{1,q}$. Since $\chi^j \in W^{1,q}_{per}(Y)$ and ϕ_j have zero average over Y, we apply a Poincaré inequality to obtain

$$\|\phi_j\|_{1,q,Y} \leq c|curl_y \phi_j|_{0,q,Y} \leq c(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y}).$$

For the case $d = 3$, the Remark 3.11 [24] guaranties the existence of $\phi_j \in W^{1,q}_{per}(Y)^3$. Since $\chi^j \in W^{1,q}_{per}(Y)$ and $u_0 \in W^{2,p}(\Omega)$ for $1/p + 1/q \leq 1/2$, we obtain $v_1(x, x/\epsilon) \in L^2(\Omega)$ and is estimated by $\|v_1\|_0 \leq c\sum_{j=1}^d \|\chi^j\|_{1,q,Y}\|u_0\|_{2,p}, d = 2, 3$. Moreover, by Lemma 3.1, we have

$$\nabla_x \cdot v_1(x, y) = 0, \tag{3.6}$$

and using elementary calculations we obtain

$$\begin{aligned}\nabla_y \cdot v_1(x,y) &= \nabla_y \cdot curl_x\left(\phi_j(y)\partial_{x_j}u_0(x)\right)\\ &= -\nabla_x \cdot curl_y\left(\phi_j(y)\partial_{x_j}u_0(x)\right)\\ &= -\nabla_x \cdot v_0(x,y) - f. \end{aligned}\tag{3.7}$$

Let

$$z_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon)$$

and

$$\eta_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon). \tag{3.8}$$

We have

$$\begin{aligned}& a(x/\epsilon)\nabla z_\epsilon(x) - \eta_\epsilon(x)\\ &\quad = a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x,x/\epsilon)\\ &\qquad - a(x/\epsilon)\nabla_y u_1(x,x/\epsilon) - a(x/\epsilon)\nabla u_\epsilon(x) + v_0(x,x/\epsilon) + \epsilon v_1(x,x/\epsilon)\\ &\quad = \epsilon(v_1(x,x/\epsilon) - a(x/\epsilon)\nabla_x u_1(x,x/\epsilon)),\end{aligned}$$

and using the notation $a^\epsilon(x) = a(x/\epsilon)$ we obtain

$$\|a^\epsilon \nabla z_\epsilon - \eta_\epsilon\|_0 \le \epsilon \|v_1(\cdot, \cdot/\epsilon) - a^\epsilon(\cdot)\nabla_x u_1(\cdot, \cdot/\epsilon)\|_0. \tag{3.9}$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H_0^1(\Omega)$ be the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall \psi \in H_0^1(\Omega). \tag{3.10}$$

Hence, using (2.6), we obtain

$$\begin{aligned}\int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon)dx &= \int_\Omega a^\epsilon \nabla w_\epsilon \cdot \nabla(z_\epsilon - \epsilon\theta_\epsilon)dx\\ \int_\Omega a^\epsilon \nabla w_\epsilon \cdot \nabla z_\epsilon dx - \epsilon\int_\Omega a^\epsilon\nabla w_\epsilon \cdot \nabla\theta_\epsilon dx &= \int_\Omega a^\epsilon \nabla w_\epsilon \cdot \nabla z_\epsilon dx.\end{aligned}\tag{3.11}$$

Now observe that

$$\int_\Omega a^\epsilon \nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a^\epsilon\nabla w_\epsilon \cdot (\nabla z_\epsilon - \eta_\epsilon)dx + \int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx. \tag{3.12}$$

In order to estimate the second term on the right-hand side of (3.12) we apply the definition of η_ϵ, see (3.8), to obtain

$$\begin{aligned}\int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon)) \cdot \nabla w_\epsilon(x)dx\\ &= \int_\Omega f w_\epsilon dx - \int_\Omega (v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon)) \cdot \nabla w_\epsilon(x)dx.\end{aligned}\tag{3.13}$$

We note that

$$\begin{aligned}&\int_\Omega v_1(x,x/\epsilon)\cdot\nabla w_\epsilon(x)dx = \int_\Omega \nabla\cdot v_1(x,x/\epsilon)w_\epsilon(x)dx =\\ &\int_\Omega (\nabla_x + 1/\epsilon\nabla_y)\cdot v_1(x,y)|_{(y=x/\epsilon)}w_\epsilon(x)dx = -\frac{1}{\epsilon}\int_\Omega(\nabla_x \cdot v_0 + f)w_\epsilon dx,\end{aligned}\tag{3.14}$$

where we have used (3.6) and (3.7) to obtain (3.14). Using the definition of v_0, see (3.3), we have

$$\int_\Omega v_0(x,x/\epsilon)\cdot\nabla w_\epsilon(x)dx = \int_\Omega a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\frac{\partial u_0}{\partial x_j}(x)\cdot\nabla w_\epsilon(x)dx,$$

and by the chain rule we obtain

$$\begin{aligned}\int_\Omega v_0(x,x/\epsilon)\cdot\nabla w_\epsilon dx &= \int_\Omega a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\cdot\nabla\left(\frac{\partial u_0}{\partial x_j}w_\epsilon(x)\right)dx \\ &\quad - \int_\Omega a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\cdot\left(w_\epsilon\nabla\frac{\partial u_0}{\partial x_j}(x)\right)dx.\end{aligned} \tag{3.15}$$

In this paragraph we evaluate the first term on the right-hand side of (3.15). Let $(\frac{\epsilon}{3}Y_i)_{i=1,\ldots,i_m}$ be a finite set of translated cells of $\frac{\epsilon}{3}Y$, covering $\overline{\Omega}$, and consider a partition of unity ρ_i, such that $\mathrm{supp}\rho_i \subset \frac{2\epsilon}{3}Y_i$, where $\frac{2\epsilon}{3}Y_i$ denotes the cell $\frac{2\epsilon}{3}Y$ centered at $\frac{\epsilon}{3}Y_i$. We note that

$$\mathrm{supp}(\rho_i w_\epsilon) \subset \frac{2\epsilon}{3}Y_i \cap \overline{\Omega} \subset \epsilon Y_i \tag{3.16}$$

thus,

$$\begin{aligned}&\int_\Omega a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\cdot\nabla(\frac{\partial u_0}{\partial x_j}w_\epsilon(x))dx = \\ &\sum_{i=1:i_m}\int_{\epsilon Y_i} a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\cdot\nabla(\rho_i\frac{\partial u_0}{\partial x_j}w_\epsilon(x))dx = 0.\end{aligned} \tag{3.17}$$

Here we have used that for Lipschitz domains, the function u_0 defined in (2.4) has a stable extension to $W^{2,p}(\mathbb{R}^d)$, see [43], which we have also denoted such extension by u_0. We also note that from (3.16) it follows that the function $\rho_i\partial_{x_j}u_0w_\epsilon$ is well defined and vanishes outside of Ω. In addition, since $1/p + 1/q \le 1/2$, we have $\rho_i\partial_{x_j}u_0w_\epsilon \in W^{1,q'}(\mathbb{R}^d)$ for $1/q' = 1 - 1/q$. In addition, $\chi^j \in W^{1,q}_{per}(Y)$ and (2.2) imply that

$$\int_Y a_{lm}(y)\partial_{y_l}(\chi^j - y_j)\partial_{y_m}\psi = 0, \quad \forall\,\psi \in W^{1,q'}_{per}(Y).$$

Therefore, since $\rho_i\partial_{x_j}u_0w_\epsilon$ has a compact support contained in the interior of ϵY_i, see (3.16), then $\rho_i\partial_{x_j}u_0w_\epsilon \in W^{1,q'}_{per}(\epsilon Y_i)$ and (3.17) follows.

For the second term on the right-hand side of equation (3.15), we use the definition of v_0, see (3.3), and $1/p + 1/q \le 1/2$, to obtain

$$-\int_\Omega a(x/\epsilon)(e_j - \nabla_y\chi^j(x/\epsilon))\cdot\left(w_\epsilon\nabla\frac{\partial u_0}{\partial x_j}(x)\right)dx = -\int_\Omega \nabla_x\cdot v_0(x,x/\epsilon)w_\epsilon(x)dx.$$

Hence,

$$\int_\Omega v_0(x,x/\epsilon)\cdot\nabla w_\epsilon(x)dx = -\int_\Omega \nabla_x\cdot v_0(x,x/\epsilon)w_\epsilon(x)dx. \tag{3.18}$$

From the equations (3.13), (3.14) and (3.18) we obtain

$$\int_\Omega \eta_\epsilon\cdot\nabla w_\epsilon dx = 0,$$

and from (3.12)

$$\int_\Omega a^\epsilon \nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a^\epsilon (\nabla z_\epsilon - \eta_\epsilon) \cdot \nabla w_\epsilon) dx. \tag{3.19}$$

Also from the equations (3.11) and (3.19) we have

$$\begin{aligned}\left| \int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon) dx \right| &\le c \|a^\epsilon \nabla z_\epsilon - \eta_\epsilon\|_0 \|w_\epsilon\|_1 \\ &\le \epsilon \|v_1(\cdot, \cdot/\epsilon) - a^\epsilon \nabla_x u_1(\cdot, \cdot/\epsilon)\|_0 \|g\|_{-1} \quad \text{by (3.9)}.\end{aligned}$$

Dividing by $\|g\|_{-1}$ and taking the supremum for $g \neq 0$, the estimate (3.1) follows since

$$\|z_\epsilon(x) - \epsilon\theta_\epsilon\|_1 \le c\epsilon \|v_1(\cdot, \cdot/\epsilon) - a^\epsilon \nabla_x u_1(\cdot, \cdot/\epsilon)\|_0 \le c\epsilon \max_j \|\chi^j\|_{1,q,Y} \|u_0\|_{2,p}.$$

We now prove the estimate (3.2). The proof is based on the maximum principle; see also [7] for a proof based on L^p boundary data. Since $u_0 \in W^{2,p}(\Omega)$ and $p > d$ we have $\partial_{x_i} u_0 \in C(\Omega)$, and since $\chi^j \in W^{1,q}_{per}(Y)$ and $q > d$ we obtain $\chi^j \in C(Y)$; see (2.5) and (2.6). Using a Sobolev embedding theorem we obtain

$$\|u_1\|_{0,\infty,\partial\Omega} \le c \max_j \|\chi^j\|_{0,\infty,Y} \|u_0\|_{1,\infty} \le c \max_j \|\chi^j\|_{1,q,Y} \|u_0\|_{2,p},$$

and using the maximum principle in [42], we have

$$\|\theta_\epsilon\|_0 \le c \|\theta_\epsilon\|_{0,\infty,\partial\Omega} = c \|u_1\|_{0,\infty,\partial\Omega} \le c \max_j \|\chi^j\|_{1,q,Y} \|u_0\|_{2,p}. \tag{3.20}$$

The estimate (3.2) follows from (3.20) and (3.1).
□

The following remark is used in the proof of the Proposition 3.5.

REMARK 3.1. *Assume that the solution u_0 of*

$$-\nabla . A \nabla u_0 = f \quad in \quad \Omega, \qquad u_0 = g \quad on \quad \partial\Omega,$$

belongs to $W^{2,p}(\Omega)$. And let $u_\epsilon \in H^1(\Omega)$ be the weak solution of

$$L_\epsilon u_\epsilon = f \quad in \quad \Omega, \quad u_\epsilon = g \quad on \quad \partial\Omega.$$

Then it is easy to see that the Proposition 3.1 extends immediately to the nonhomogeneous Dirichlet boundary condition case.

The Proposition 3.2 estimates the H^1 norm of $\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon$ in terms of a parameter r; see Remark 3.2 for the discussion on r. The Proposition 3.2 is used in the proof of the Theorems 3.1 and 3.2.

Proposition 3.2. *Let $\Omega = (0,1)^2$ be the unit square. Let u_0, $\tilde{\theta}_\epsilon$ and $\tilde{\phi}_\epsilon$ be defined by the equations (2.4), (2.10) and (2.14), respectively. Assume that $a_{ij} \in L^\infty_{per}(Y)$ and $u_0 \in W^{2,p}(\Omega)$. Let the functions v_k be defined as in Subsection 2.2.1. Assume that $exp(-\gamma y_1)\nabla v_e \in L^{\tilde{q}}(G_e)$ and $exp(-\gamma y_1)(v_e - \chi^*_e) \in L^{\tilde{q}}(G_e)$, and similar conditions also for the other functions v_k, $k \in \{w, n, s\}$. Let l be defined by $\frac{1}{p} + \frac{1}{\tilde{q}} + \frac{1}{l} = 1$, $1 \le p, \tilde{q} \le \infty$. Let r be any number such that $r > 2$. Then, there is positive constant $c = c(\gamma, l, r)$ and $\delta = \delta(p, l, r) = -\frac{1}{p} + \frac{1}{lr}$ such that*

$$\begin{aligned}\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_1 \le c\epsilon^\delta \max_k \big(&\|exp(-\gamma y \cdot \eta^k)\nabla v_k\|_{0,\tilde{q},G_k} \\ &+ \|exp(-\gamma y \cdot \eta^k) v_k - \chi^*_k\|_{1,\tilde{q},G_k}\big) \|u_0\|_{2,p}.\end{aligned}$$

In addition, when $p,q=\infty$ then $\delta=\frac{1}{2}$, when $\frac{3}{p}+\frac{1}{\tilde{q}}<1$ then there exists an $r>2$ such that $\delta>0$, and when $\frac{1}{p}+\frac{1}{\tilde{q}}<\frac{1}{2}$ then there exists an $r>2$ such that $\delta>-\frac{1}{p}+\frac{1}{4}$.

Proof. By definition

$$\|\tilde{\theta}_\epsilon-\tilde{\phi}_\epsilon\|_1\leq\sum_{k\in\{e,w,n,s\}}\|\tilde{\theta}^k_\epsilon-\tilde{\phi}^k_\epsilon\|_1.$$

Consider the case $k=e$, the other cases are treated similarly. Denote $a^\epsilon(x)=a(x/\epsilon)$, let $v^\epsilon_e(x)=v_e(\frac{x_1-1}{\epsilon},\frac{x_2}{\epsilon})$ and take $g\in H^1_0(\Omega)$. Then applying the definition of $\tilde{\phi}^e_\epsilon$ and $\tilde{\theta}^e$, see (2.13) and (2.12) respectively, we obtain

$$\begin{aligned}\int_\Omega a^\epsilon\nabla(\tilde{\theta}^e_\epsilon-\tilde{\phi}^e_\epsilon)\cdot\nabla g dx&=\int_\Omega -a^\epsilon\nabla\left((v^\epsilon_e-\chi^*_e)\varphi_e\frac{\partial u_0}{\partial x_1}\right)\cdot\nabla g dx\\ &=-\int_\Omega\left(\varphi_e\frac{\partial u_0}{\partial x_1}a^\epsilon\nabla v^\epsilon_e\right)\cdot\nabla g dx-\int_\Omega\left((v^\epsilon_e-\chi^*_e)a^\epsilon\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}\right)\right)\cdot\nabla g dx.\end{aligned}\tag{3.21}$$

We note that due to the Sobolev embedding Theorem 5.4 [2], the integrals above are well defined. For estimating the first term on the right hand-side of the equation (3.21) we let

$$\begin{aligned}&\int_\Omega\left(\varphi_e\frac{\partial u_0}{\partial x_1}a^\epsilon\nabla v^\epsilon_e\right)\cdot\nabla g dx=\\ &\qquad\int_\Omega a^\epsilon\nabla v^\epsilon_e\cdot\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}g\right)dx-\int_\Omega a^\epsilon\nabla v^\epsilon_e\cdot g\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}\right)dx.\end{aligned}\tag{3.22}$$

We now estimate the first term of the right-hand side of (3.22). Let $I_i=\{(i-1)\epsilon/6-\epsilon/6<x_2<i\epsilon/6+\epsilon/6,\}$, $i_{max}=1+\sup_{i\in\mathbb{N}}(i3/\epsilon<1)$, and consider a partition of unity ρ_i of Ω, subject to $(0,1)\times I_i$. Let I^ϵ_i be the interval centered in I_i with $|I^\epsilon_i|=\epsilon$. Since $\text{supp}(\rho_i g)\subset[0,1]\times I^\epsilon_i$ we have

$$\begin{aligned}&\int_\Omega a^\epsilon\nabla v^\epsilon_e\cdot\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}g\right)dx=\\ &\qquad\sum_{i=0:i_{max}}\int_0^1\int_{I^\epsilon_i}a^\epsilon\nabla v^\epsilon_e\cdot\nabla\left(\rho_i\varphi_e\frac{\partial u_0}{\partial x_1}g\right)dx_2dx_1=0,\end{aligned}\tag{3.23}$$

where to arrive to (3.23) we have used the definition of v_e and arguments similar to the ones given to obtain (3.17).

For estimating the second term on the right-hand side of the equation (3.22) we apply Hölder's inequality to obtain

$$\begin{aligned}&\left|\int_\Omega a^\epsilon\nabla v^\epsilon_e\cdot\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}\right)g dx\right|\leq\\ &\|a\|_\infty|\varphi_e\nabla u_0|_{1,p}\left\|\exp(-\gamma\frac{x_1-1}{\epsilon})\nabla v^\epsilon_e\right\|_{0,\tilde{q}}\left(\frac{\epsilon}{\gamma}\right)^{1/l}\left\|(\gamma/\epsilon)^{1/l}\exp(\gamma\frac{x_1-1}{\epsilon})g\right\|_{0,l},\end{aligned}\tag{3.24}$$

where $1/l = 1 - 1/p - 1/\tilde{q}$. Taking $y_1 = (x_1-1)/\epsilon$ and $y_2 = x_2/\epsilon$, and exploring the $[0,1]$-periodicity of $v_e(y_1,\cdot)$ we have

$$\left\|\nabla(v_e^\epsilon)\exp(-\gamma\frac{x_1-1}{\epsilon})\right\|_{0,\tilde{q}} \le \left((\frac{1}{\epsilon}+1)\int_{-1/\epsilon}^{0}\int_0^1 |\exp(-\gamma y_1)\nabla_y v_e|^s \epsilon^{2-\tilde{q}} dy_2 dy_1\right)^{\frac{1}{\tilde{q}}}$$

$$\le c\epsilon^{(\frac{1}{\tilde{q}}-1)}\|\exp(-\gamma y_1)\nabla_y v_e\|_{0,\tilde{q},G_e}. \tag{3.25}$$

Now let $g_n \in C_0^\infty(\Omega)$ such that $g_n \to g$ in H^1, and let $I_n = (0,1)\cap|g_n|>0$. Using integration by parts in x_1 we obtain

$$\left\|(\gamma/\epsilon)^{1/l}\exp(\gamma\frac{x_1-1}{\epsilon})g_n\right\|_{0,l} = \left(\int_0^1\int_{I_n}\frac{\gamma}{\epsilon}\exp(l\gamma\frac{x_1-1}{\epsilon})|g_n|^l dx_1 dx_2\right)^{1/l}$$

$$= \left(-\int_0^1\int_{I_n}\frac{1}{l}\exp\left(l\gamma\frac{x_1-1}{\epsilon}\right)\frac{\partial|g_n|^l}{\partial x_1}dx_1dx_2\right)^{1/l} \tag{3.26}$$

$$\le c\left(\left\|\exp\left(l\gamma\frac{x_1-1}{\epsilon}\right)\right\|_{0,r}\|g_n\|_{0,s(l-1)}^{l-1}\left\|\frac{\partial g_n}{\partial x_1}\right\|_0\right)^{1/l} \tag{3.27}$$

$$\le c(s(l-1))^{(l-1)/l}\left(\frac{\epsilon}{rl\gamma}\right)^{1/(rl)}|g_n|_1, \tag{3.28}$$

where we have used a Hölder's inequality with $1/s+1/r = 1/2, s<\infty$ to obtain (3.27). To obtain (3.28), we use the following inequality used in the proof of Lemma 5.10 [2], i.e., there exists $1 \le t < 2$ such that

$$\|g_n\|_{0,r(l-1)} \le 2^{(t-1)/t}\left(\frac{2t-t}{2-t}\right)\|g_n\|_{1,t}, \quad \text{for } 2t/(2-t) = s(l-1)$$

$$\le 2^{(t-1)/t}\left(\frac{2t-t}{2-t}\right)\text{vol}(\Omega)^{(1/t-1/2)}\|g_n\|_1, \text{ by Theorem 2.8 [2]}$$

$$\le c\left(\frac{2t-t}{2-t}\right)|g_n|_1, \text{ by a Poincaré inequality.}$$

Hence, (3.28) follows from (3.27). Now taking the limit $n\to\infty$ in (3.28), we obtain the inequality (3.28) for g. Thus, using (3.22), (3.23), (3.24), (3.25) and (3.28), it follows that

$$\int_\Omega \varphi_e\frac{\partial u_0}{\partial x_1}a^\epsilon\nabla v_e^\epsilon\cdot\nabla g dx \le c(\gamma)(s(l-1))^{(l-1)/l}\epsilon^\delta\|a\|_\infty|\varphi_e\nabla u_0|_{1,p}$$

$$\|\exp(-\gamma y_1)\nabla v_e\|_{0,\tilde{q},G_e}|g|_1, \tag{3.29}$$

where $\delta = -\frac{1}{p}+\frac{1}{lr}$. When $p,\tilde{q}\to\infty$ and then $r\to 2$, then $(s(l-1))^{(l-1)/l}(\epsilon/(rl\gamma))^{1/(rl)} \to \epsilon^{1/2}/(2\gamma)$. Also it is easy to see that $\frac{3}{p}+\frac{1}{\tilde{q}}<1$ implies that there exists $r<2$ such that $\delta = -\frac{1}{p}+\frac{1}{rl}>0$, and $\frac{1}{p}+\frac{1}{\tilde{q}}<\frac{1}{2}$ implies that there exists $r<2$ such that $\delta = -\frac{1}{p}+\frac{1}{rl}>-\frac{1}{4}$.

For estimating the second term on the right-hand side of (3.21), we apply a Hölder's inequality to obtain

$$\left|\int_\Omega(v_e^\epsilon-\chi_e^*)a^\epsilon\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}\right)\cdot\nabla g dx\right|$$

$$\leq \|a\|_{0,\infty} \left|\varphi_e \frac{\partial u_0}{\partial x_1}\right|_{1,p} \left\|(v_e^\epsilon - \chi_e^*)\exp(-\gamma \frac{x_1-1}{\epsilon})\right\|_{0,\tilde{q}} \left\|\exp(\gamma \frac{x_1-1}{\epsilon})\right\|_{0,l}$$

$$\leq c(\gamma)\epsilon^{\frac{1}{2}-\frac{1}{p}}\|a\|_{0,\infty} \left|\varphi_e \frac{\partial u_0}{\partial x_1}\right|_{1,p} \|(v_e - \chi_e^*)\exp(-\gamma y_1\|_{0,\tilde{q},G_e} |g|_1, \tag{3.30}$$

and we note that $\delta < -\frac{1}{p} + \frac{1}{2}$ since $0 \leq \frac{1}{l} \leq 1$ and $2 < r$.

Taking $g = \tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e$ and using the ellipticity of a, we obtain

$$\begin{aligned}|\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e|^2_{H_0^1(\Omega)} &\leq \gamma_a^{-1} \int_\Omega (a^\epsilon \nabla(\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e)) \cdot \nabla(\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e) dx \\ &\leq c\epsilon^\delta \|a\|_{0,\infty} |\varphi_e \nabla u_0|_{1,p} (\|\nabla(v_e - \chi_e^*)\exp(-\gamma y_1)\|_{0,\tilde{q},G_e} \\ &\quad + \|(v_e - \chi_e^*)\exp(-\gamma y_1\|_{0,\tilde{q},G_e}).\end{aligned}$$

□

REMARK 3.2. *When $r \to 2$ then $\delta(p,l,r) = \frac{1}{p} - \frac{1}{rl}$ increases to $\frac{1}{p} - \frac{1}{2l}$. We note however that $r \to 2$ implies that $s \to \infty$ and therefore, the a priori constant $c(\gamma,l,r) \to \infty$ if $l \neq 1$; see the term $(s(l-1))^{(l-1)/l}(\frac{\epsilon}{r\gamma l})^{\frac{1}{rl}}$ in (3.28). The optimal choice of r (denoted by r^*) to minimize this term depends on ϵ, l and weakly on γ. When $l \to 1$ then $r^*(l) \to 2$, therefore, $\delta \to -\frac{1}{p} + \frac{1}{2}$. Also, it is easy to see that when $\epsilon \to 0$ then $r^* \to 2$. Hence, for the worst case, i.e. $p = 2$ and $\tilde{q} = \infty$, when ϵ is very small then δ is very close to $-\frac{1}{4}$.*

We next prove the last proposition used in the proof of the Theorem 3.1. The Proposition 3.3 estimates the H^1 norm of $\bar{\phi} - \bar{\theta}_\epsilon$.

Proposition 3.3. *Let $\Omega = (0,1)^2$ be the unit square. Let the functions u_0, $\bar{\theta}_\epsilon$ and $\bar{\phi}$ be defined by the equations (2.4), (2.11) and (2.15), respectively. Assume that $a_{ij} \in L^\infty_{per}(Y)$ and $u_0 \in H^2(\Omega)$. Then there exists a positive constant c, independent of ϵ, u_0, and χ^j, such that*

$$\|\bar{\phi} - \bar{\theta}_\epsilon\|_1 \leq c \max_j \|\chi^j\|_{1,2,Y} \|u_0\|_2.$$

Proof. Using that $(\bar{\phi} - \bar{\theta}_\epsilon) = 0$ on $\partial\Omega$ we have

$$\begin{aligned}\int_\Omega a_{ij}^\epsilon \frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_i} \frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j} dx &= \int_\Omega a_{ij}^\epsilon \frac{\partial \bar{\phi}}{\partial x_i} \frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j} dx \\ &\leq \|a\|_{0,\infty,Y} \left(\int_\Omega |\nabla \bar{\phi}|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla(\bar{\phi} - \bar{\theta}_\epsilon)|^2 dx\right)^{1/2},\end{aligned}$$

and from the ellipticity of a and using Remark 2.1 we obtain

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq \frac{\|a\|_{0,\infty,Y}}{\gamma_a} |\bar{\phi}|_1 \leq \tilde{c} \frac{\|a\|_{0,\infty,Y}}{\gamma_a} \max_j \|\chi^j\|_{1,2,Y} \|u_0\|_2.$$

The proposition follows from a Poincaré inequality.
□

The following proposition generalizes the Proposition 2.3 [35], where it is assumed that $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^3(\Omega)$ and $\Omega \subset \mathbb{R}^2$ is a smooth domain.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$ be a bounded domain with Lipschitz boundary. Let u_ϵ be the solution of the problem (1.1), and let χ^j, u_0, u_1, θ_ϵ, χ^{ij}, u_2*

and φ_ϵ *be defined by the equations (2.2), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9), respectively. Assume that* $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, χ^j *and* $\chi^{ij} \in W^{1,q}_{per}(Y)$ *for* $1/p + 1/q \leq 1/2$. *Then there exists a constant c independent of* u_0, ϵ, χ^j *and* χ^{ij} *such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot) - \epsilon^2 u_2(\cdot, \cdot/\epsilon) - \epsilon^2\varphi_\epsilon(\cdot)\|_1 \leq C\epsilon^2(\max_j \|\chi^j\|_{0,q,Y} + \max_{kj} \|\chi^{kj}\|_{1,q,Y})\|u_0\|_{3,p}.$$

In addition, if $p, q > d$ *then*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_0 \leq c\epsilon^2(\max_j \|\chi^j\|_{0,q} + \max_{kj} \|\chi^{kj}\|_{1,q})\|u_0\|_{3,p}.$$

Proof.
Define the field v_1 by

$$(v_1(x,y))_k = -a_{ki}(y)\chi^j \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + a_{kl}(y)\frac{\partial \chi^{ij}}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x), \tag{3.31}$$

then

$$a(y)\nabla_x u_1(x,y) + a(y)\nabla_y u_2(x,y) = v_1(x,y). \tag{3.32}$$

Let $q(y) = \phi(y)$, where ϕ is defined by the equation (3.5), and let $\psi_{ij} \in W^{1,q}_{\text{per}}(Y)$ such that

$$curl_y \psi_{1j} = \tilde{\psi}_{1j} = \begin{pmatrix} -a_{11}\chi^j + a_{1l}\partial_l\chi^{1,j} - c^1_{1j} \\ -a_{21}\chi^j + a_{2l}\partial_l\chi^{1,j} - \phi^{(3)}_j - c^2_{1j} \\ -a_{31}\chi^j + a_{3l}\partial_l\chi^{1,j} + \phi^{(2)}_j - c^3_{1j} \end{pmatrix},$$

$$curl_y \psi_{2j} = \tilde{\psi}_{2j} = \begin{pmatrix} -a_{12}\chi^j + a_{1l}\partial_l\chi^{2,j} + \phi^{(3)}_j - c^1_{2j} \\ -a_{22}\chi^j + a_{2l}\partial_l\chi^{2,j} - c^2_{2j} \\ -a_{32}\chi^j + a_{3l}\partial_l\chi^{2,j} - \phi^{(1)}_j - c^3_{2j} \end{pmatrix},$$

and

$$curl_y \psi_{3j} = \tilde{\psi}_{3j} = \begin{pmatrix} -a_{13}\chi^j + a_{1l}\partial_l\chi^{3,j} - \phi^{(2)}_j - c^1_{3j} \\ -a_{23}\chi^j + a_{2l}\partial_l\chi^{3,j} + \phi^{(1)}_j - c^2_{3j} \\ -a_{33}\chi^j + a_{3l}\partial_l\chi^{3,j} - c^3_{3j} \end{pmatrix}.$$

Here the constants c^l_{ij} are chosen so that each entry of the vectors $\tilde{\psi}_{ij}$ has zero value integral over Y; for instance, $c^1_{1j} = \int_Y -a_{11}\chi^j + a_{1l}\partial_l\chi^{1,j}dy$. It is easy to check that $\nabla_y \cdot \tilde{\psi}_{kj} = 0$, hence, the existence of such functions ψ_{kj} is guaranteed by Lemma 3.1, and by the Remark 3.11 [24]. We then obtain

$$\|\psi_{kj}\|_{1,q} \leq c(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{3.33}$$

Define

$$p(x,y) = \psi_{kj}(y)\frac{\partial^2 u_0}{\partial x_k \partial x_j}(x) \tag{3.34}$$

and let

$$v_2(x,y) = -curl_x p(x,y).$$

An elementary calculation gives

$$\nabla_y \cdot v_2 = -\nabla_x \cdot v_1, \quad \nabla_x \cdot v_2 = 0 \tag{3.35}$$

and

$$\begin{aligned} \|v_2(\cdot,\cdot/\epsilon)\|_0 &\le c\|u_0\|_{3,p} \max_{kj} \|\psi_{kj}\|_{1,q,Y} \\ &\le c\|u_0\|_{3,p}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}) \quad \text{by} \quad (3.33). \end{aligned} \tag{3.36}$$

Define

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x,x/\epsilon) - \epsilon^2 u_2(x,x/\epsilon)$$

and

$$\xi_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon) - \epsilon^2 v_2(x,x/\epsilon),$$

where v_0 is defined by (3.3). Then

$$\begin{aligned} a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon(x) &= a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla u_0(x) - \epsilon a(x/\epsilon)\nabla u_1(x,x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla u_2(x,x/\epsilon) \\ &\quad -a(x/\epsilon)\nabla u_\epsilon(x) + v_0(x,x/\epsilon) + \epsilon v_1(x,x/\epsilon) + \epsilon^2 v_2(x,x/\epsilon) \\ &= -a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x,x/\epsilon) - a(x/\epsilon)\nabla_y u_1(x,x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla_x u_2(x,x/\epsilon) - \epsilon a(x/\epsilon)\nabla_y u_2(x,x/\epsilon) \\ &\quad +v_0(x,x/\epsilon) + \epsilon v_1(x,x/\epsilon) + \epsilon^2 v_2(x,x/\epsilon) \\ &= \epsilon^2(v_2(x,x/\epsilon) - a(x/\epsilon)\nabla_x u_2(x,x/\epsilon)), \quad \text{by (3.3) and (3.32).} \end{aligned}$$

Using (3.36) and the definition of u_2, see (2.8), we obtain

$$\|a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon\|_0 \le c\epsilon^2\|u_0\|_{3,p} \max_{kj}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{3.37}$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H^1(\Omega)$ denote the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall\psi \in H_0^1(\Omega). \tag{3.38}$$

Since $\psi_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\varphi_\epsilon \in H_0^1(\Omega)$ we obtain

$$\begin{aligned} \int_\Omega g(\psi_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\varphi_\epsilon)dx &= \int_\Omega a(x/\epsilon)(\nabla\psi_\epsilon - \epsilon\nabla\theta_\epsilon - \epsilon^2\nabla\varphi_\epsilon)\nabla w_\epsilon(x)dx \\ &= \int_\Omega a(x/\epsilon)\nabla\psi_\epsilon\nabla w_\epsilon(x)dx, \end{aligned} \tag{3.39}$$

where we have used the definition of θ_ϵ and φ_ϵ to obtain (3.39). We observe that

$$\int_\Omega a^\epsilon\nabla\psi_\epsilon\nabla w_\epsilon dx = \int_\Omega (a^\epsilon\nabla\psi_\epsilon - \xi_\epsilon)\cdot\nabla w_\epsilon dx + \int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon dx, \tag{3.40}$$

and we estimate the second term on the right-hand side of (3.40) as follows:

$$\begin{aligned}\int_\Omega \xi_\epsilon \cdot \nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon) \\ &\quad -\epsilon^2 v_2(x,x/\epsilon)) \cdot \nabla w_\epsilon(x) dx \\ &= \int_\Omega f(x)w_\epsilon(x) + \nabla_x \cdot v_0(x,x/\epsilon)w_\epsilon(x) \\ &\quad -\epsilon v_1(x,x/\epsilon)\cdot \nabla w_\epsilon(x) + \epsilon \nabla_x v_1(x,x/\epsilon) w_\epsilon(x) dx, \end{aligned} \tag{3.41}$$

where we have used the definition of u_ϵ, (3.18), integration by parts, and (3.35) to obtain (3.41). Using (3.31) we have

$$\begin{aligned}\int_\Omega v_1(x,x/\epsilon)\cdot\nabla w_\epsilon(x) &= \int_\Omega \left(-a^\epsilon_{ki}\chi^j_\epsilon \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) \right. \\ &\quad \left. + a^\epsilon_{kl}\frac{\partial \chi^{ij}_\epsilon}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\right)\frac{\partial w_\epsilon}{\partial x_k}(x)dx.\end{aligned} \tag{3.42}$$

Considering the partition of unit ρ_i defined in the proof of the Proposition 3.1, we obtain

$$\begin{aligned}\int_\Omega a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}(x)dx &= \sum_1^{i_m}\int_{\epsilon Y_i} a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}dx \\ = \sum_1^{i_m}\int_{\epsilon Y_i} a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\frac{\partial}{\partial x_k}&\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}w_\epsilon(x)\right) - a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}w_\epsilon(x)\frac{\partial}{\partial x_k}\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}\right)dx \\ = \sum_1^{i_m}\int_{\epsilon Y_i}\epsilon^{-1}&\left(a^\epsilon_{ij} - a^\epsilon_{ik}\frac{\partial\chi^j_\epsilon}{\partial_{y_k}} + A_{ij}\right)\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}w_\epsilon \\ +a^\epsilon_{ki}\chi^j_\epsilon\left(\frac{\partial}{\partial_{x_k}}\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\right)\right.&\left. w_\epsilon(x) + \rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\frac{\partial w_\epsilon}{\partial_{x_k}}(x)\right)dx \\ &- \int_\Omega a_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}w_\epsilon(x)\frac{\partial}{\partial x_k}\left(\frac{\partial^2 u_0}{\partial x_j\partial x_i}\right)dx\end{aligned} \tag{3.43}$$

$$\begin{aligned} = \int_\Omega \epsilon^{-1}&\left(\nabla_x v_0\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x) - f\right)w_\epsilon(x)dx \\ &- \int_\Omega a^\epsilon_{ki}\chi^j_\epsilon\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial_{x_k}}(x)dx - \int_\Omega \nabla_x\cdot v_1 dx.\end{aligned} \tag{3.44}$$

Here, we have used the definition of χ^{ij} to arrive to (3.43). From (3.41), (3.42) and (3.44) we obtain

$$\int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon(x)dx = 0,$$

and therefore, from (3.37) and (3.40) we obtain

$$\begin{aligned}\left|\int_\Omega g(\psi_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\varphi_\epsilon)dx\right| &\leq \|a^\epsilon\nabla\psi_\epsilon - \xi_\epsilon)\|_0\|w_\epsilon\|_1 \\ &\leq c\epsilon^2\|u_0\|_{3,p}(\|\chi^j\|_{0,q,Y} + \|\chi^{kj}\|_{1,q,Y})\|g\|_{-1}.\end{aligned}$$

Dividing by g and taking the supremum over g, (3.31) follows since

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon - \epsilon^2 u_2 - \epsilon^2\varphi_\epsilon\|_1 \leq c\epsilon^2\|u_0\|_{3,p} \max_{kj}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}).$$

The remaining of the proof uses the same arguments given to prove (3.2).
□

The following proposition estimates the L^2 norm of $\bar{\phi} - \bar{\theta}_\epsilon$, and it is used in the proof of the Theorem 3.2.

Proposition 3.5. *Let $\Omega = (0,1)^2$ be the unit square. Let u_0, χ^j, $\bar{\theta}_\epsilon$ and $\bar{\phi}$ be defined by (2.4), (2.2), (2.11) and (2.15), respectively. Assume that $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$ for $1/p + 1/q \leq 1/2$, $2 \leq p, q \leq \infty$. Then we have*

$$\|\bar{\theta}_\epsilon - \bar{\phi}\|_0 \leq c\epsilon \max_j \|\chi^j\|_{1,2,Y}\|u_0\|_{3,p}.$$

Proof. Using the arguments given in Remark 2.1 we have

$$\|\sum_k \varphi_k \chi^*_k \nabla u_0 \cdot \eta_k\|_{2,p} \leq c \max_k |\chi^*_k| \|u_0\|_{3,p} \leq c \max_j \|\chi^j\|_{1,2,Y}\|u_0\|_{3,p} \tag{3.45}$$

where we have used that $|\chi^*_k| \leq c \max_j \|\chi^j\|_{1,2,Y}$; see the proof of Lemma 4.4 [3]. By assumption, $u_0 \in H^1_0(\Omega) \cap W^{3,p}(\Omega)$. Using a trace theorem and the Remarks 2.1 and 5.1, we obtain that

$$\bar{\phi}|_{\partial\Omega} = \sum_k \varphi_k \chi^*_k \nabla u_0 \cdot \eta_k|_{\partial\Omega}$$

belongs to $W^{2-\frac{1}{p},p}(\Gamma_k) \cap W^{1-\frac{1}{p},p}_{00}(\Gamma_k)$, $k = \{e, w, n, s\}$. Following Chapter 5 [25], when Ω is a is a polygonal convex domain then the problem is $W^{2,p}(\Omega)$ regular, and therefore, the solution of (2.16) belongs to $W^{2,p}(\Omega)$ and satisfies $\|\bar{\phi}\|_{2,p} \leq c \max_j \|\chi^j\|_{1,2,Y}\|u_0\|_{3,p}$. From the Remark 3.1 and (3.2) we obtain

$$\|\bar{\theta}_\epsilon - \bar{\phi}\|_0 \leq c\epsilon \max_j \|\chi^j\|_{1,2,Y}\|\bar{\phi}\|_{2,p}$$

and the proposition follows.
□

The following proposition is used in the proof of the Propositions 3.1 and 3.4.

Lemma 3.1. *A function $\mathbf{v} \in L^2_{per}(Y)^2$, ($\mathbf{v} \in L^2_{per}(Y)^3$) satisfies*

$$\nabla \cdot \mathbf{v} = \mathbf{0}, \tag{3.46}$$

and $\int_Y v_i dy = 0$ iff there exists a function $\phi \in H^1_{per}(Y)$ ($\phi \in H^1_{per}(Y)^3$) such that:

$$\mathbf{v} = curl\phi. \tag{3.47}$$

Proof. The proof is similar to the proof of Theorem 3.4 [24], where here we use discrete Fourier transforms rather than continuous Fourier transforms, see [44]. □

The following theorems provide error estimates between u_ϵ and $u_0 - \epsilon u_1 - \epsilon\phi_\epsilon$ on the H^1 and L^2 norms.

Theorem 3.1. *Let $\Omega = (0,1)^2$ be the unit square and u_ϵ be the solution of the problem (1.1). Let u_0, u_1, ϕ_ϵ be defined by equations (2.4), (2.5) and (2.16),* *respectively. Assume* $a_{ij} \in L^\infty_{per}(Y)$*,* $u_0 \in W^{2,p}(\Omega)$*, and* $\chi^j \in W^{1,q}_{per}(Y)$ *for* $1/p + 1/q \leq 1/2$*. Let the functions* v_k *be defined as in Subsection 2.2.1. Assume that* $exp(-\gamma y_1)\nabla v_e \in L^{\tilde{q}}(G_e)$ *and* $exp(-\gamma y_1)(v_e - \chi^*_e) \in L^{\tilde{q}}(G_e)$*, for* $\frac{1}{p} + \frac{1}{\tilde{q}} \leq \frac{1}{2}$*, and similar conditions also for the other functions* v_k*,* $k \in \{w, n, s\}$*. Define* $\hat{\delta} = \min\{0, \delta\}$*,* δ *defined by the Proposition 3.2 and Remark 3.2. Then there exists a constant* c *independent of* ϵ *such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_1 \leq c\epsilon^{1+\hat{\delta}}\|u_0\|_{2,p}.$$

In addition, if $\frac{1}{p} + \frac{1}{\tilde{q}} < \frac{1}{2}$ *then* $\hat{\delta} > -\frac{1}{p} + \frac{1}{4}$*, and if* $\frac{3}{p} + \frac{1}{\tilde{q}} < 1$ *then* $\hat{\delta} = 0$*.*

Proof. By the triangular inequality we have

$$|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\phi_\epsilon|_1 \leq |u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon|_1 \\ +\epsilon|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon|_1 + \epsilon|\bar{\theta}_\epsilon - \bar{\phi}|_1,$$

and applying the Propositions 3.1, 3.2 and 3.3, and the Remark 3.2, the theorem follows.

□

REMARK 3.3. *Theorem 1 [34] guaranties that* $\chi^j \in W^{1,q}_{per}(Y)$*, for all* $q < \infty$*. See also Theorem 1.1 [31] for conditions on* a_{ij} *in order to have* $\chi^j \in W^{1,\infty}_{per}(Y)$*.* The following proposition gives higher error estimates when more regularity is assumed.

Theorem 3.2. *Let* $\Omega = (0,1)^2$ *be the unit square and* u_ϵ *be the solution of the problem (1.1). Let* u_0*,* u_1*,* ϕ_ϵ *be defined by the equations (2.4), (2.5) and (2.16), respectively. Assume* $u_0 \in W^{3,p}(\Omega)$*,* $\bar{\phi} \in W^{2,p}(\Omega)$*,* χ^j *and* $\chi^{ij} \in W^{1,q}_{per}(Y)$ *for* $1/p + 1/q \leq 1/2$*,* $p, q > 2$*. Let the functions* v_k *be defined as in Subsection 2.2.1. Assume that* $exp(-\gamma y_1)\nabla v_e \in L^\infty(G_e)$ *and* $exp(-\gamma y_1)(v_e - \chi^*_e) \in L^\infty(G_e)$*, and similar conditions also for the other functions* v_k*,* $k \in \{w, n, s\}$*. Then there exists a constant* c *independent of* ϵ *such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_0 \leq c\epsilon^{3/2}\|u_0\|_{3,p}.$$

Proof. Use a triangular inequality similar to the one used in the proof of the Theorem 3.1, use a Sobolev embedding theorem to have $u_0 \in W^{2,\infty}(\Omega)$, and then apply the Propositions 3.4, 3.2 and 3.5. We obtain the factor $O(\epsilon^{3/2})$ rather than the (ϵ^2) as in Proposition 3.4 since $\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_0$ is $O(\epsilon^{1/2})$; see the Remark 3.2 and the Proposition 3.2.

□

REMARK 3.4. *Note that if* $a_{ij} \in C^{1,\beta}_{per}(Y)$ $\beta > 0$*, then from regularity theory we have* $\chi^j \in C^{1,\beta}_{per}$*,* $v_e \in C^{1,\beta}$ *and* $\nabla(v_e - \chi^*_e)exp(-\gamma y_1) \in L^\infty(G_e)$*; see Theorem 15.1 [29] and Remark 6.4 [35]; see also Theorem 3 [30] for related issues.*

4. Finite Element Approximation. For the Sections 4 and 5 we assume that A, χ^j, v_k and χ^* are computed exactly. We now describe the finite element approximations for u_0, u_1, $\tilde{\phi}_\epsilon$ and $\bar{\phi}$.

- Let $V^h(\Omega)$ be the P_1 or the Q_1 conforming finite element space on the mesh $\mathcal{T}_h(\Omega)$ and let $V^h_0(\Omega) = V^h(\Omega) \cap H^1_0(\Omega)$. Define $u^h_0 \in V^h_0(\Omega)$ as the solution of

$$\int_\Omega A\nabla u^h_0 \cdot \nabla v^h dx = \int_\Omega f v^h dx, \quad \forall v^h \in V^h_0(\Omega). \tag{4.1}$$

- We now introduce the discrete approximation for $\partial_\eta u_0$. Note that $\partial_\eta u_0$ is used on the boundary condition imposed for problem (2.15). In order to approximate $\partial_\eta u_0$ we define $Y^h = V^h(\Omega)|_{\partial\Omega}$, $Y^h_k = Y^h|_{\Gamma_k}$ and $Y^h_{0,k} = \{\lambda^h \in Y^h_k; \lambda^h = 0 \text{ at } \partial\Gamma_k\}$. For all $\phi^h \in V^h(\Omega)$ and $\phi^h|_{\partial\Omega\setminus\Gamma_k} = 0$, let $\lambda^h_k \in Y^h_{0,k}$ be defined as the solution of

$$\int_{\Gamma_k} \lambda^h_k \phi^h d\sigma = \int_\Omega A\nabla u^h_0 \cdot \nabla\phi^h dx - \int_\Omega f\phi^h dx; \tag{4.2}$$

  see [48]. Later in the Lemma 5.3 we show that λ^h_k is a good approximation for $A\nabla u_0 \cdot \eta_k$ on Γ_k, hence, we approximate $\partial_\eta u_0$ by μ^h where

$$\mu^h|_{\Gamma_k} = \lambda^h_k / A_{l_k l_k},\ l_k = \begin{cases} 1 & \text{if} \quad k = e, w \\ 2 & \text{if} \quad k = n, s. \end{cases} \tag{4.3}$$

- We note that we use μ^h as the approximation for $\partial_\eta u_0$ in the equation (4.7). Therefore, in order to guarantee that the final numerical approximation for u_ϵ satisfy the zero Dirichlet boundary condition, we define the approximation for ∇u_0 as

$$\Psi^h = \nabla u^h_0 + \sum_{k\in\{e,w,n,s\}} E^h_k(\mu^h - \nabla u^h_0 \cdot \eta^k)\eta^k. \tag{4.4}$$

  Here $E^h_k(\cdot)$ denotes the trivial zero non-conforming discrete extension of $\mu^h - \nabla u^h_0 \cdot \eta^k$ in Ω. More specifically, $E^h_k(\mu^h - \nabla u^h_0 \cdot \eta^k)(z) = 0$ if z is nodal point of $\mathcal{T}_h(\overline{\Omega})\setminus\Gamma_k$, $E^h_k(\mu^h - \nabla u^h_0 \cdot \eta^k) = \mu^h - \nabla u^h_0 \cdot \eta^k$ on Γ_k, and $E^h_k(\mu^h - \nabla u^h_0 \cdot \eta^k)$ inside each element $K_i \in \mathcal{T}_h(\Omega)$ belongs to $V_h(K_i) := V^h(\Omega)|_{K_i}$.
- Define

$$u^h_1(x, x/\epsilon) = -\Psi^h_j(x)\chi^j(x/\epsilon). \tag{4.5}$$

  Note that this leads to a nonconforming approximation for u_1 in $\mathcal{T}_h(\Omega)$.
- Define

$$\tilde\phi^h_\epsilon = \sum_{k\in\{e,w,n,s\}} \tilde\phi^{k,h}_\epsilon, \tag{4.6}$$

  where

$$\tilde\phi^{e,h}_\epsilon(x_1, x_2) = \varphi_e(x_1, x_2)\left(v_e\left(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon}\right) - \chi^*_e\right)\Psi^h,$$

  see (2.13) and (4.4), and the others terms $\tilde\phi^{k,h}_\epsilon$ are defined similarly.
- Let $\bar\phi^h \in V_h(\Omega)$ be the solution of

$$\int_\Omega A\nabla\bar\phi^h \cdot \nabla v^h dx = 0, \quad \forall v^h \in V^h_0(\Omega), \quad \text{and} \quad \bar\phi^h = \chi^*\mu^h \quad \text{on} \quad \partial\Omega. \tag{4.7}$$

  The well posedness of (4.7) follows from Remark 4.1.
- Approximate ϕ_ϵ by $\phi^h_\epsilon = \tilde\phi^h_\epsilon + \bar\phi^h$ and finally define the numerical solution for the equation (1.1) as

$$u^h_\epsilon = u^h_0 + \epsilon u^h_1 + \epsilon\phi^h_\epsilon. \tag{4.8}$$

REMARK 4.1. *By construction μ^h vanishes at the corners of Ω, therefore, $\chi^*\mu^h \in H^{1/2}(\partial\Omega)$. This implies that the equation (4.7) is well posed. In addition, $\chi^*\mu^h \in V^h|_{\partial\Omega}$, hence, we can look for a numerical solution of the equation (4.7) in $V^h(\Omega)$.*

5. Finite Element Error Analysis. The main results of this section are the Theorems 5.1 and 5.2 which provide the a priori error estimates between the exact solution u_ϵ and its numerical approximation u_ϵ^h. The Theorem 5.1 provides the a priori error estimates on the broken H^1-norm, while the Theorem 5.2 provides on the L_2-norm.

In this section we recall few results on discrete Sobolev norms and spaces, prove the propositions and lemmas required in the proofs of the Theorems 5.1 and 5.2, and then we close the section proving the Theorems 5.1 and 5.2.

5.1. Preliminaries. We first review general finite element results that are used later in this section. Let u_0 and u_0^h be the solutions of the problems (2.4) and (4.1), respectively. We consider $\mathcal{P}_1$ or $\mathcal{Q}_1$ finite elements spaces for approximating u_0. We assume that the domain Ω is a convex polygon, therefore, the problem (2.4) is $W^{2,p}$ regular for some $2 < p$; see [25]. Standard finite element analysis provides

$$\|u_0 - u_0^h\|_{1,p} \leq ch\|u_0\|_{2,p}, \text{ for } 2 \leq p \leq \infty, \tag{5.1}$$

$$\|u_0 - u_0^h\|_{0,p} \leq ch^2\|u_0\|_{2,p}, \text{ for } 2 \leq p < \infty \tag{5.2}$$

and

$$\|u_0 - u_0^h\|_{2,p,h} \leq c\|u_0\|_{2,p}, \text{for } 2 \leq p \leq \infty; \tag{5.3}$$

see the Corollary 7.1.12, the inequality (7.5.4) and the Theorem 4.4.20 [12] to obtain (5.1), and standard duality arguments to obtain (5.2). To obtain the stability result (5.3) we first introduce the interpolation operator $\mathcal{I}^h$, i.e., the usual local point-wise interpolation $\mathcal{P}_1$ or $\mathcal{Q}_1$ in $\mathcal{T}_h(\Omega)$. Then, for each element $K \in \mathcal{T}_h(\Omega)$, we have

$$|u_0 - u_0^h|_{2,p,K} \leq |u_0 - \mathcal{I}^h u_0|_{2,p,K} + |\mathcal{I}^h u_0 - u_0^h|_{2,p,K}, \tag{5.4}$$

and using an interpolation error estimate, see Theorem 4.4.20 [12], we obtain

$$|u_0 - \mathcal{I}^h u_0|_{s,p,h} \leq ch^{2-s}|u_0|_{2,p,h}, \text{ for } 1 \leq p \leq \infty, \text{ for } 0 \leq s \leq 2. \tag{5.5}$$

Now apply an inverse inequality, see Lemma 4.5.3 [12], to obtain

$$|\mathcal{I}^h u_0 - u_0^h|_{2,p,K} \leq ch^{-1}\|\mathcal{I}^h u_0 - u_0^h\|_{1,p,K}, \text{ for } 1 \leq p \leq \infty, \tag{5.6}$$

and hence, (5.3) follows from (5.4), (5.5), (5.6) and (5.1).

In order to estimate the L^2- and the broken H^1- norms of $u_1 - u_1^h$, see the Proposition 5.1, we apply Hölder's inequality on $u_1 - u_1^h = (\partial_{x_j} u_0 - \Psi_j^h)\chi^j$ to obtain $\|u_1 - u_1^h\|_{1,h} \leq c\|\partial_{x_j} u_0 - \Psi_j^h\|_{1,p,h}\|\chi^j\|_{1,q}$ and $\|u_1 - u_1^h\|_0 \leq c\|\partial_{x_j} u_0 - \Psi_j^h\|_{0,p}\|\chi^j\|_{0,q}$, for $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ and. Hence, it requires estimating the error between Ψ^h and ∇u_0 on the L^p-norm and the broken $W^{1,p}$-norm; see the Lemma 5.4. The Lemma 5.4 requires estimating the errors between $A\nabla u_0 \cdot \eta$ and λ^h, see (4.2), over Γ_k using fractional-order Sobolev norms.

REMARK 5.1. *Here, we review some facts about fractional-order Sobolev norms and spaces that are used throughout the text; see [25, 33].*

Case $2 < p < \infty$**:** *Since* $W^{1-\frac{1}{p},p}(\Gamma_k) \hookrightarrow C^0(\Gamma_k)$, *we define the spaces* $W_{00}^{1-\frac{1}{p},p}(\Gamma_k) = \{\varphi \in W^{1-\frac{1}{p},p}(\Gamma_k);\ \varphi = 0 \text{ on } \partial\Gamma_k\}$ *equipped with the norm* $\|\cdot\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} = \|\cdot\|_{W^{1-\frac{1}{p},,p}(\Gamma_k)}$.

Case $p=2$**:** *We set* $W_{00}^{1-\frac{1}{p},p}(\Gamma_k)=H_{00}^{1/2}(\Gamma_k)$ *equipped with the norm* $\|\cdot\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}=\|\cdot\|_{H_{00}^{1/2}(\Gamma_k)}$*; see [33] for the definition of* $H_{00}^{1/2}(\Gamma_k)$*.*

Case $1<p<2$**:** *We define* $W_{00}^{1-\frac{1}{p},p}(\Gamma_k)=W^{1-\frac{1}{p},p}(\Gamma_k)$ *equipped with the norm* $\|\cdot\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}=\|\cdot\|_{W^{1-\frac{1}{p},p}(\Gamma_k)}$.

We note that for Lipschitz domains, we have that $W_{00}^{1-\frac{1}{p},p}(\Gamma_k)$ *are also equivalent to* $[L^p(\Gamma_k),W_0^{1,p}(\Gamma_k)]_{1-1/p,p}$*, i.e., via real interpolation theory on Banach spaces; see [25]. This property is used throughout the text to establish stabilities and a priori error estimates results.*

The spaces $W_{00}^{1-\frac{1}{p},p}(\Gamma_k)$ *have also the following important feature. Denote by* $\tilde{\varphi}$ *the extension by zero to* $\partial\Omega\setminus\Gamma_k$ *of a given function* $\varphi\in W_{00}^{1-1/p,p}(\Gamma_k)$*. Then, by the Trace Theorem and the Lift Theorem 1.5.2.3 [25] there exists a function* $\psi_\varphi\in W^{1,p}(\Omega)$ *such that* $\psi_\varphi|_{\partial\Omega}=\tilde{\varphi}$ *and*

$$c_1\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\le\|\psi_\varphi\|_{1,p}\le c_2\|\tilde{\varphi}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\le c_3\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}. \tag{5.7}$$

The dual space of $W_{00}^{1-\frac{1}{p},p}(\Gamma_k)$ *is denoted by* $W^{-1+\frac{1}{p},p'}(\Gamma_k)$*, where* $\frac{1}{p}+\frac{1}{p'}=1$.

5.2. Finite Element Convergence Analysis. We now prove the lemmas and propositions required to complete the proof of the Theorems 5.1 and 5.2.

The following inverse inequality is required in the proof of the Lemma 5.3.

Lemma 5.1. *Let* $1<p<\infty$ *and* $v^h\in Y_{0,k}^h$*. Then*

$$\|v^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\le ch^{-1}\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)}. \tag{5.8}$$

Proof. Considering the inverse inequality given by the Theorem 4.5.11 [12] followed by the real interpolation method, see the Proposition 12.1.5 [12], we have

$$\|v^h\|_{s,q,\partial\Omega}\le ch^{-s}\|v^h\|_{0,q,\partial\Omega},\quad\forall\, v^h\in Y^h,\ 1\le p\le\infty \text{ and } 0\le s\le 1. \tag{5.9}$$

Given $v^h\in Y_{0,k}^h$ let $\tilde{v}^h\in Y^h$ be the extension of v^h to $\partial\Omega\setminus\Gamma_k$ by zero. Using (5.7) and (5.9) we obtain

$$\begin{aligned}\|v^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}&\le c\|\tilde{v}^h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\\ &\le ch^{-1+\frac{1}{p}}\|\tilde{v}^h\|_{L^p(\partial\Omega)}=ch^{-1+\frac{1}{p}}\|v^h\|_{L^p(\Gamma_k)}.\end{aligned} \tag{5.10}$$

Let $\mathcal{P}_{0,k}$ denote the L^2 projection on $Y_{0,k}^h$ and assume that $v^h\in Y_{0,k}^h$. Then

$$\|v^h\|_{L^p(\Gamma_k)}=\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle v^h,\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}=\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle v^h,\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}.$$

We also have by Theorem 1 [17] that

$$\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}\le c\|\phi\|_{L^{p'}(\Gamma_k)}\quad 1\le p'\le\infty. \tag{5.11}$$

Thus,

$$\begin{aligned}\|v^h\|_{L^p(\Gamma_k)}&\le c\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}}\\ &\le ch^{-1+\frac{1}{p'}}\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)},\end{aligned} \tag{5.12}$$

where on the last inequality we have used (5.10). Combining inequalities (5.10) and (5.12) we obtain (5.8).

□

The following lemma provides stability and approximation results associated to $\mathcal{P}_{0,k}$, i.e., the L^2 projection on $Y^h_{0,k}$. These results are required in the proof of the Lemma 5.3.

Lemma 5.2. *Let $1 < p < \infty$ and $\mathcal{P}_{0,k} : W^{-1+\frac{1}{p},p'}(\Gamma_k) \to Y^h_{0,k}$ be the L^2 projection on $Y^h_{0,k}$. Then we have*

$$\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \leq c\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\, \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{5.13}$$

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \leq ch^{1-\frac{1}{p}}\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\, \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{5.14}$$

$$\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \leq c\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \quad \forall\, \phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k) \tag{5.15}$$

and

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \leq ch^{1-\frac{1}{p}}\|\phi\|_{L^{p'}(\Gamma_k)} \quad \forall\, \phi \in L^{p'}(\Gamma_k). \tag{5.16}$$

Proof of (5.13): Observe that $\mathcal{P}_{0,k} : L^p(\Gamma_k) \to Y^h_{0,k}$ is stable in L^p and $W_0^{1,p}$, $1 \leq p \leq \infty$, i.e., $\|\mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \leq c\|\phi\|_{L^p(\Gamma_k)}$ $\forall\phi \in L^p(\Gamma_k)$, and $\|\mathcal{P}_{0,k}\phi\|_{W^{1,p}(\Gamma_k)} \leq c\|\phi\|_{W^{1,p}(\Gamma_k)}$ $\forall\phi \in W_0^{1,p}(\Gamma_k)$, respectively; see Theorems 1 and 2 of [17]. Since $W_{00}^{1-1/p,p}(\Gamma_k)$ is equivalent to $[L^p(\Gamma_k), W_0^{1,p}(\Gamma_k)]_{1-1/p,p}$, the stability of $\mathcal{P}_{0,k}$ in the norm $W_{00}^{1-1/p,p}(\Gamma_k)$ follows from the real interpolation method; see the Proposition 12.1.5 [12]. The inequality (5.13) then follows.

Proof of (5.14): Let $\mathcal{Q}^h : L^p(\Gamma_k) \to V^h(\Omega)|_{\Gamma_k}$ denote the $\mathcal{P}_1$ or $\mathcal{Q}_1$ Clement interpolation operator defined by (2.13) in [41]. Then we have

$$\begin{aligned} \|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} &\leq \|\phi - \mathcal{Q}^h\phi\|_{L^p(\Gamma_k)} + \|\mathcal{P}_{0,k}(\phi - \mathcal{Q}^h\phi)\|_{L^p(\Gamma_k)} \\ &\leq c\|\phi - \mathcal{Q}^h\phi\|_{L^p(\Gamma_k)}, \text{ by } (5.11) \\ &\leq ch^{1-\frac{1}{p}}\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}, \end{aligned} \tag{5.17}$$

where to obtain (5.17) we have used the Theorem 4.1 and the Lemma 4.1 [41] followed by the real interpolation method.

Proofs of (5.15) and (5.16) It follows from standard duality arguments, the properties of the L_2 projection $\mathcal{P}_{0,k}$ and the estimates (5.13) and (5.14), respectively.

□

The following lemma estimates the error between $\lambda = A\nabla u_0 \cdot \eta$ and its numerical approximation λ^h defined by the equation (4.2). This lemma is used in the proof of the Lemma 5.4.

Lemma 5.3. *Assume that $u_0 \in W^{2,p}(\Omega)$. Then we have*

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}, \|\lambda - \lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \leq c\|u_0\|_{2,p} \quad \text{for} \;\; 2 \leq p < \infty, \tag{5.18}$$

$$\|\lambda - \lambda^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)} \le ch\|u_0\|_{2,p} \quad \textit{for} \quad 2 \le p < \infty. \tag{5.19}$$

and

$$\|\lambda - \lambda^h\|_{L^p(\Gamma_k)} \le ch^{1-\frac{1}{p}}\|u_0\|_{2,p} \quad \textit{for} \quad 2 \le p \le \infty. \tag{5.20}$$

Proof of (5.18): From Remark 2.1 and a trace theorem we have

$$\|\lambda\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le c\|u_0\|_{2,p}. \tag{5.21}$$

In order to prove the inequality (5.18) observe that

$$\|\lambda - \lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le \|\lambda\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} + \|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}, \tag{5.22}$$

and

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} = \sup_{\phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle \lambda^h, \phi\rangle}{\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}.$$

Since $\lambda^h \in Y_{0,k}^h$ then $\langle \lambda^h, \phi\rangle = \langle \lambda^h, \mathcal{P}_{0,k}\phi\rangle$, and using (5.15) we obtain

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le c \sup_{\phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle \lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}. \tag{5.23}$$

Let $E : W^{-1+\frac{1}{p},p'}(\partial\Omega) \to W^{1,p'}(\Omega)$ be a stable lift extension and let $\mathcal{Q}_h$, see [41], be the modified Clement L_2-quasi-projection satisfying the discrete boundary condition. Then $E_h = \mathcal{Q}_h E : Y^h \to V^h(\Omega)$, see (5.5) in [41], satisfies

$$\|E_h g\|_{1,p'} \le c\|g\|_{W^{1-\frac{1}{p'},p'}(\partial\Omega)}$$

for $g \in Y^h$. Hence, if $g^h \in Y_{0,k}^h$ and $\tilde{g}^h$ denotes the extension of g^h by zero to $\partial\Omega \setminus \Gamma_k$ it follows

$$\|E_h \tilde{g}^h\|_{1,p'} \le c\|g^h\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}. \tag{5.24}$$

Let $\tilde{\mathcal{P}}_{0,k}\phi$ denote the discrete extension of $\mathcal{P}_{0,k}\phi$ to $\partial\Omega \setminus \Gamma_k$ by zero. From the definition of λ^h, see (4.2), the inequalities (5.1) and (5.24), and the inverse inequality (5.8), we obtain

$$\begin{aligned}
\langle \lambda^h, \mathcal{P}_{0,k}\phi\rangle &= \langle \lambda, \mathcal{P}_{0,k}\phi\rangle + \int_\Omega A\nabla(u_0^h - u_0)\cdot\nabla(E_h\tilde{\mathcal{P}}_{0,k}\phi)dx \\
&\le \|\lambda\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} + ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)} \\
&\le \left(\|\lambda\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} + c\|u_0\|_{2,p}\right)\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}.
\end{aligned} \tag{5.25}$$

The inequality (5.18) follows from (5.23), (5.25), (5.22) and (5.21).

Proof of (5.19): We observe that

$$\|\lambda-\lambda^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)} = \sup_{\phi\in W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)} \frac{\langle\lambda-\lambda^h,\phi\rangle}{\|\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}}$$

$$\leq \sup_{\phi\in W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)} c\frac{\langle\lambda-\lambda^h,\mathcal{P}_{0,k}\phi\rangle}{\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}} + \sup_{\phi\in W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)} c\frac{\langle\lambda-\lambda^h,\phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}}. \tag{5.26}$$

In order to estimate the first term on the right-hand side of (5.26) we use the definition of λ and λ^h, and the inequality (5.24) to obtain

$$\begin{aligned}\langle\lambda-\lambda^h,\mathcal{P}_{0,k}\phi\rangle &= \int_\Omega A\nabla(u_0^h-u_0)\cdot\nabla(E_h\tilde{\mathcal{P}}_{0,k}\phi)dx \\ &\leq ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}.\end{aligned} \tag{5.27}$$

For estimating the second term on the right-hand side of (5.26) we use (5.18) and (5.16) to obtain

$$\begin{aligned}\langle\lambda-\lambda^h,\phi-\mathcal{P}_{0,k}\phi\rangle &\leq \|\lambda-\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\|\phi-\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \\ &\leq ch\|u_0\|_{2,p}\|\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)},\end{aligned} \tag{5.28}$$

and the inequality (5.19) follows from (5.26), (5.27) and (5.28).

Proof of (5.20):

Case $2\leq p<\infty$: We have

$$\|\lambda-\lambda^h\|_{L^p(\Gamma_k)} \leq \sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle\lambda-\lambda^h,\phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} + \sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle\lambda-\lambda^h,\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}. \tag{5.29}$$

The first term on the right-hand side of (5.29) is bounded as follows:

$$\begin{aligned}\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle\lambda-\lambda^h,\phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\leq \sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\|\lambda-\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\|\phi-\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}{\|\phi\|_{L^{p'}(\Gamma_k)}} \\ &\leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}.\end{aligned} \tag{5.30}$$

Here we have used (5.16) and (5.18) to arrive to (5.30). In order to estimate the second term on the right hand-side of (5.29) we use the definition of λ and λ^h to obtain

$$\begin{aligned}\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\langle\lambda-\lambda^h,\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\leq c\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\int_\Omega A\nabla(u_0-u_0^h)\cdot\nabla(E_h\tilde{\mathcal{P}}_{0,k}\phi)dx}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\ &\leq ch\sup_{\phi\in L^{p'}(\Gamma_k)}\frac{\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\ &\leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}, \text{ by (5.10).}\end{aligned}$$

Case $p = \infty$: Let $z \in \Gamma_k$, then

$$|\lambda(z) - \lambda^h(z)| \leq |\lambda(z) - \mathcal{P}_{0,k}\lambda(z)| + |\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)|. \tag{5.31}$$

To estimate the first term of the right-hand side of (5.31), we use that the L^2 projection is stable in any L^p norm, $1 \leq p \leq \infty$; see Lemma 3.5 [47].

To estimate the second term on the right-hand side of (5.31) we use the techniques developed in [47]. Let $E_z \subset \Gamma_k$ denote an edge of an element $K_z \in \mathcal{T}^h(\Omega)$ such that $z \in E_z$, and define δ_z as the polynomial of degree 1 on E_z such that

$$\int_{E_z} \delta_z(s)v(s)ds = v(z), \text{ for any } v \text{ polynomial of degree 1.}$$

Regard δ_z as extended by zero to $\Gamma_k \setminus E_z$ and denote by $\tilde{\delta}_z^h \in V^h(\Omega)$ the extension by zero of $\mathcal{P}_{0,k}\delta_z$ to Ω. Then we have

$$\begin{aligned} \lambda^h(z) - \mathcal{P}_{0,k}\lambda(z) &= \int_{\Gamma_k} \mathcal{P}_{0,k}(\lambda^h - \lambda)\delta_z ds = \int_{\Gamma_k} (\lambda^h - \lambda)\mathcal{P}_{0,k}\delta_z ds \\ &= \int_\Omega A\nabla(u_0 - u_0^h) \cdot \nabla\tilde{\delta}_z^h dx \end{aligned} \tag{5.32}$$

where we have used the definition of λ^h to obtain (5.32). From (5.1) and (5.32) it follows that

$$|\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)| \leq ch\|u_0\|_{2,\infty}\|\tilde{\delta}_z^h\|_{1,1}.$$

Using an inverse inequality followed by a discrete Sobolev inequality for trivial extensions, we have

$$\|\tilde{\delta}_z^h\|_{1,1} \leq ch^{-1}\|\tilde{\delta}_z^h\|_{0,1} \leq c\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k}.$$

Finally, we use the fact that $\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k} \leq c$, see Lemma 3.5 [47], to obtain (5.20). □

The next lemma estimates the error between ∇u_0 and its numerical approximation Ψ^h. This proposition is required in the proof of the Proposition 5.1.

Lemma 5.4. *Let u_0 and Ψ^h be defined by the equations (2.4) and (4.4), respectively. Assume that $u_0 \in W^{2,p}(\Omega)$ for $2 \leq p \leq \infty$. Then*

$$\|(\nabla u_0 - \Psi^h) \cdot \nu\|_{0,p} \leq ch\|u_0\|_{2,p}, \quad \forall\, \nu \in \mathbb{R}^2 \text{ with } |\nu| = 1 \tag{5.33}$$

and

$$\|(\nabla u_0 - \Psi^h) \cdot \nu\|_{1,p,h} \leq c\|u_0\|_{2,p}, \quad \forall\, \nu \in \mathbb{R}^2 \text{ with } |\nu| = 1. \tag{5.34}$$

Proof of (5.33): From the triangular inequality we have

$$\|(\nabla u_0 - \Psi^h) \cdot \nu\|_{0,p} \leq \|(\nabla u_0 - \nabla u_0^h) \cdot \nu\|_{0,p} + \|(\nabla u_0^h - \Psi^h) \cdot \nu\|_{0,p}. \tag{5.35}$$

The first term on the right-hand side of (5.35) is estimated by (5.1), while for the second term we use (4.4) to obtain

$$\|(\nabla u_0^h - \Psi^h) \cdot \nu\|_{0,p} \leq c \sum_{k\in\{e,w,n,s\}} \|E_k^h(\mu^h - \nabla u_0^h \cdot \eta^k)\|_{0,p}.$$

Consider the case $k = e$ and $\mathcal{Q}_1$ conforming finite elements for approximating u_0; the other cases, i.e., $k \in \{w, n, s\}$ or $\mathcal{P}_1$ conforming finite elements, can be treated similarly. The function $E_e^h\left(\mu^h - \frac{\partial u_0^h}{\partial x_1}\right)$ is piecewise linear in the x_1 direction and equal to zero for $x_1 \leq 1-h$; see the definition of E_e^h in (4.4). Hence, using a discrete Sobolev inequality for trivial extensions we obtain

$$\|E_e^h(\mu^h - \nabla u_0^h \cdot \eta^k)\|_{0,p} \leq h^{\frac{1}{p}} \left\|\partial_{x_1} u_0^h - \mu^h\right\|_{0,p,\Gamma_e}, \quad \text{if} \ \ 2 \leq p < \infty$$

or

$$\|E_e^h(\mu^h - \nabla u_0^h \cdot \eta^k)\|_{0,\infty} \leq \left\|\partial_{x_1} u_0^h - \mu^h\right\|_{0,\infty,\Gamma_e}, \quad \text{if} \ \ p = \infty.$$

Case $2 \leq p < \infty$: The triangular inequality gives

$$\left\|\partial_{x_1} u_0^h - \mu^h\right\|_{0,p,\Gamma_e} \leq \left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,\Gamma_e} + \left\|\partial_{x_1} u_0 - \mu^h\right\|_{0,p,\Gamma_e}. \tag{5.36}$$

In order to estimate the first term on the right-hand side of (5.36), let K be an element of $\mathcal{T}_h(\Omega)$ containing an edge $E \subset \Gamma_k$. Applying a trace theorem we have

$$\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,E} \leq$$
$$c\left(h^{-1}\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,K}^p + h^{p-1}\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\|_{1,p,K}^p\right)^{\frac{1}{p}}. \tag{5.37}$$

From (5.37), (5.1) and (5.3) we obtain

$$\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,\Gamma_e} \leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}. \tag{5.38}$$

We now estimate the second term on the right-hand side of (5.36). Let $\lambda = A\nabla u_0 \cdot \eta$ and λ_h and μ_h defined by (4.2) and (4.3), respectively. Using that the tangential derivate of u_0 vanish on $\partial\Omega$, we obtain

$$\partial_{x_1} u_0 - \mu^h = A_{11}(\lambda - \lambda^h) \tag{5.39}$$

hence, from (5.20) we have

$$\|\partial_{x_1} u_0 - \mu^h\|_{0,p,\Gamma_e} \leq A_{11}\|\lambda - \lambda^h\|_{0,p,\Gamma_e} \leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}. \tag{5.40}$$

From (5.36), (5.38) and (5.40) we obtain

$$\|E_e(\mu^h - \nabla u_0^h \cdot \eta^e)\|_{0,p} \leq ch\|u_0\|_{2,p},$$

and thus the estimate (5.33) holds for $2 \leq p \leq \infty$.

Case $p = \infty$: We have

$$\left\|\partial_{x_1} u_0^h - \mu^h\right\|_{0,\infty,\Gamma_e} \leq \|\partial_{x_1} u_0^h - \partial_{x_1} u_0\|_{0,\infty,\Gamma_e} + \|\partial_{x_1} u_0 - \mu^h\|_{0,\infty,\Gamma_e},$$

and applying (5.1), (5.39) and (5.20) we have

$$\left\|\partial_{x_1} u_0 - \mu^h\right\|_{0,\infty,\Gamma_e} \leq ch\|u_0\|_{2,\infty},$$

and hence, estimate (5.33) follows for $p=\infty$.

Proof of (5.34): We have

$$\begin{aligned} \|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p} &\le c\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{0,p} + \|(\nabla u_0 - \nabla u_0^h)\cdot\nu\|_{0,p} \\ &\le ch\|u_0\|_{2,p}, \quad \text{by } \ (5.1) \text{ and } (5.33), \end{aligned} \tag{5.41}$$

and from an inverse inequality, see Lemma 4.5.3 [12], it follows that

$$\|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{1,p,h} \le c\|u_0\|_{2,p}. \tag{5.42}$$

Since

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{1,p,h} \le c\left(\|(\nabla u_0^h - \nabla u_0)\cdot\nu\|_{1,p,h} + \|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{1,p,h}\right),$$

we obtain (5.34) from (5.3) and (5.42).

□

The following proposition is required in the proofs of the Theorems 5.1 and 5.2.

Proposition 5.1. *Let u_1 and u_1^h be defined by (2.5) and (4.5), respectively. Assume that $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$, for $\frac{1}{p}+\frac{1}{q}\le\frac{1}{2}$, $2\le p,q\le\infty$. Then there exists a constant c independent of ϵ and h such that*

$$|u_1 - u_1^h|_{1,h} \le c\left(\frac{h^2}{\epsilon^2}+1\right)^{1/2} \max_j \|\chi^j\|_{1,q,Y}\|u_0\|_{2,p} \tag{5.43}$$

and

$$\|u_1 - u_1^h\|_0 \le ch \max_j \|\chi^j\|_{1,q,Y}\|u_0\|_{2,p}. \tag{5.44}$$

Proof of (5.43): We have

$$\begin{aligned} &|u_1 - u_1^h|_{1,h}^2 \le \\ &\quad 2\sum_{K\in\mathcal{T}_h(\Omega)}\int_K \sum_{i\in 1,2}((\partial_{x_j}u_0 - \Psi_j^h)\partial_{x_i}\chi^j(\cdot/\epsilon))^2 + (\chi^j(\cdot/\epsilon)\cdot\partial_{x_i}(\partial_{x_j}u_0 - \Psi_j^h))^2 dx. \end{aligned} \tag{5.45}$$

For the first term on the right-hand side of (5.45) we have

$$\begin{aligned} \sum_{K\in\mathcal{T}_h(\Omega)}\int_K((\partial_{x_j}u_0 - \Psi_j^h)\partial_{x_i}\chi^j(\cdot/\epsilon))^2 dx &\le |\partial_{x_j}u_0 - \Psi_j^h|_{0,p}^2\|\partial_{x_i}\chi^j(\cdot/\epsilon)\|_{0,q}^2 \\ &\le \epsilon^{-2}|\partial_{x_j}u_0 - \Psi_j^h|_{0,p}^2\|\chi^j\|_{1,q,Y}^2 \le c\epsilon^{-2}h^2\|u_0\|_{2,p}^2\|\chi^j\|_{1,q,Y}^2, \end{aligned} \tag{5.46}$$

where we have used (5.33) to obtain (5.46).

For the second term on the right-hand side of (5.45) we use a Hölder's inequality to obtain $\|\chi^j\partial_{x_i}(\partial_{x_j}u_0 - \Psi_j^h)\|_0^2 \le \|\chi^j\|_{0,q}^2|\partial_{x_j}u_0 - \Psi_j^h|_{1,p,h}^2$.

Proof of (5.44): It follows from a direct application of the Hölder's inequality and the approximation error estimate (5.33).

□

The next proposition is required in the proofs of Theorems 5.1 and 5.2.

Proposition 5.2. *Let $\tilde{\phi}_\epsilon$ and $\tilde{\phi}^h_\epsilon$ be defined by the equations (2.14) and (4.6), respectively. Let the functions v_k be defined as in the Subsection 2.2.1. Assume that $u_0 \in W^{2,p}(\Omega)$ and $v_k \in W^{1,\tilde{q}}(G_k)$, for $\frac{1}{p} + \frac{1}{\tilde{q}} \leq \frac{1}{2}$, $2 \leq p, \tilde{q} \leq \infty$. Then*

$$|\tilde{\phi}_\epsilon - \tilde{\phi}^h_\epsilon|_{1,h} \leq c\left(\frac{h^2}{\epsilon^2}+1\right)^{\frac{1}{2}} \max_k \|v_k\|_{1,\tilde{q},G_k}\|u_0\|_{2,p} \tag{5.47}$$

and

$$\|\tilde{\phi}_\epsilon - \tilde{\phi}^h_\epsilon\|_0 \leq ch \max_k \|v_k - \chi^*_k\|_{0,\tilde{q},G_k}\|u_0\|_{2,p}. \tag{5.48}$$

Proof. From definition of $\tilde{\phi}_\epsilon$ and $\tilde{\phi}^h_\epsilon$ we have

$$|\tilde{\phi}_\epsilon - \tilde{\phi}^h_\epsilon|_{1,h} \leq \sum_{k\in\{e,w,n,s\}} |\tilde{\phi}^k_\epsilon - \tilde{\phi}^{k,h}_\epsilon|_{1,h},$$

and using the arguments similar to the ones given in the proof of the Proposition 5.1, the proposition follows.
□

We now prove the last proposition used in the proof of the Theorems 5.1 and 5.2.

Proposition 5.3. *Let $\bar{\phi}$ and $\bar{\phi}^h$ be defined by the equations (2.15) and (4.7) respectively, and assume that $u_0 \in H^2(\Omega)$. Then we have*

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \leq c\|u_0\|_2 \tag{5.49}$$

and

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \leq ch\|u_0\|_2. \tag{5.50}$$

Proof of (5.49): Consider the triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \leq \|\bar{\phi}^h - \psi\|_1 + \|\bar{\phi} - \psi\|_1,$$

where $\psi \in H^1(\Omega)$ is defined as the weak solution of

$$-\nabla\cdot A\nabla\psi = 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \psi = \chi^*\mu^h \quad \text{on} \quad \partial\Omega. \tag{5.51}$$

We note that the problem (5.51) is well defined since $\chi^*\mu^h \in H^{1/2}(\partial\Omega)$; see Remark 4.1. From regularity theory, (5.18), (5.39) and Remark 2.1 we have the following estimate

$$\|\psi\|_1 \leq \sum_k c\|\chi^*\mu^h\|_{H^{1/2}_{00}(\Gamma_k)} \leq c\|u_0\|_2. \tag{5.52}$$

Note also that $\chi^*\mu^h$ belongs to Y^h. Hence, using (5.52) and standard finite element analysis for conforming finite element discretizations we obtain

$$|\bar{\phi}^h - \psi|_1 \leq c\|u_0\|_2.$$

Finally, from regularity theory, (5.39) and (5.20) we obtain

$$\begin{aligned} |\bar{\phi} - \psi|_1 &\leq \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \\ &\leq \sum_k \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \leq c\|u_0\|_2. \end{aligned}$$

Proof of (5.50): From the triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \leq c\|\bar{\phi} - \psi\|_0 + \|\bar{\phi}^h - \psi\|_0,$$

and from standard finite element analysis and (5.52) we obtain

$$\|\bar{\phi}^h - \psi\|_0 \leq ch\|\psi\|_1 \leq ch\|u_0\|_2.$$

Finally, from Theorem 6.1 [38], (5.39) and (5.19) we obtain

$$\begin{aligned}\|\bar{\phi} - \psi\|_0 &\leq c(\sum_k \|\chi^*\partial_\eta u_0 - \chi^*\mu^h\|^2_{H^{-1/2}(\Gamma_k)})^{1/2} \\ &\leq ch\|u_0\|_2 \quad \text{by (5.19).}\end{aligned}$$

□

Finally, we prove the Theorems 5.1 and 5.2.

Theorem 5.1. *Assume the same conditions of the Theorem 3.1 and let* u_ϵ *be defined by (4.8). Then there exists a constant* c *independent of* ϵ, u_0 *and* h *such that*

$$|u_\epsilon - u_\epsilon^h|_{1,h} \leq c(h + \epsilon^{1+\hat{\delta}})\|u_0\|_{2,p} \tag{5.53}$$

and

$$\|u_\epsilon - u_\epsilon^h\|_0 \leq c(h^2 + \epsilon^{1+\hat{\delta}})\|u_0\|_{2,p}. \tag{5.54}$$

Proof. From the triangular inequality we have

$$\begin{aligned}|u_\epsilon - u_\epsilon^h|_{1,h} \leq |u_\epsilon - u_0 - u_1 - \phi_\epsilon|_1 + |u_0 - u_0^h|_1 + \epsilon|u_1 - u_1^h|_{1,h} \\ +\epsilon|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} + \epsilon|\bar{\phi} - \bar{\phi}^h|_1,\end{aligned}$$

and (5.53) follows from Theorem 3.1, the approximation error (5.1), and the Propositions 5.1, 5.2 and 5.3. Using similar arguments and (5.2) the estimate (5.54) follows.
□

Theorem 5.2. *Assume the same conditions of the Theorem 3.2 and let* u_ϵ *be defined by (4.8). Then there exists a constant* c *independent of* ϵ, u_0 *and* h *such that*

$$\|u_\epsilon - u_\epsilon^h\|_0 \leq c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)\|u_0\|_{3,p}.$$

Proof. Repeat the proof of the Theorem 5.1 and use the Theorem 3.2.
□

6. Numerical Results. The objective of this section is to compare the difference between u_ϵ and u_ϵ^h on the L^2-norm and on the broken H^1-semi-norm. We note however that an explicit formula for u_ϵ is not known, hence, we replace u_ϵ by a regular $\mathcal{P}_1$ conforming finite element solution of the problem (1.1) on a very fine uniform mesh of size h_f; denote this numerical solution by u_ϵ^*.

We note we have assumed in the Sections 4 and 5 that the cell and boundary layer problems were solved exactly in order to define χ, A, v_k and χ^*. In [44, 46] we have developed an accurate numerical algorithm for approximating these values, and as a result, the errors between u_ϵ^h and u_ϵ^* do not depend if the approximated or the exact values were used or not. We now describe how to approximate χ, A, v_k, χ^* and $\tilde{\phi}_\epsilon$ and display only the changes that should be made in the algorithm in the Section 4 in order to obtain the numerical results of this section.

- Replace χ^j by the finite element solution of the problem (2.2) using $\mathcal{P}_1$ or $\mathcal{Q}_1$ conforming finite element method on a partition $\mathcal{T}_{\hat{h}}(Y)$.
- Replace A_{ij} by $\frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i-\chi^i)\frac{\partial}{\partial y_m}(y_j-\chi^j)dy$; see (2.3).
- Let τ be a positive integer and $G_e^\tau = \{y \in \mathbb{R}^2;\ -\tau \le y_1 \le 0 \text{ and } 0 \le y_2 \le 1\}$. Using a $\mathcal{P}_1$ or $\mathcal{Q}_1$ conforming finite element, replace v_e by the finite element solution associated to the following problem:

$$
\begin{aligned}
&-\nabla_y \cdot a(y)\nabla_y v_e = 0 \ \text{ in } \ G_e^\tau,\\
&v_e(y) = \chi^1(1/\epsilon, y_2) \ \text{ on } \ \{y \in G_e^\tau, y_1 = 0\},\\
&\partial_\eta v_e = 0 \ \text{ on } \{y \in G_e^\tau;\ y_1 = -\tau\},\\
&\text{and } v_e(y_1, 0) = v_k(y_1, 1) \ \text{ for } \ -\tau \le y_1 \le 0,
\end{aligned}
$$

and replace χ_e^* by $\int_0^1 v_e(-\tau, y_2)dy_2$. We replace $\tilde{\phi}_\epsilon^e$ by

$$
\tilde{\phi}_\epsilon^e(x_1,x_2) = \begin{cases} \left(v_e\left(\frac{x_1-1}{\epsilon},\frac{x_2}{\epsilon}\right) - \chi_e^*\right)\Psi^h & \text{if } \ x_1 > 1-\epsilon\tau \\ 0 & \text{otherwise.}\end{cases}
$$

The other cases $k \in \{w, n, s\}$ are treated similarly.

In the numerical experiments we use $\mathcal{P}_1$ conforming finite element with a uniform mesh $\hat{h} = \frac{1}{128}$ and $\tau = 2$.

For the first test of experiments we consider a smooth coefficient $a(y_1, y_2)$ case, see [26], where

$$
a(y) = \left(\frac{2+1.8\sin(2\pi y_1)}{2+1.8\cos(2\pi y_2)} + \frac{2+\sin(2\pi y_2)}{2+1.8\sin(2\pi y_1)}\right) I_{2\times 2}, \quad \text{and} \quad f(x) = -1.
$$

The Table 6.1 shows the errors between u_ϵ^* and u_ϵ^h. We see that for $\epsilon << h$ the errors are of order h^2 and h on the L^2-norm and the on broken H^1-semi-norm, respectively. We can see that when we fix h and decrease ϵ then the errors change very little. This is an evidence that the dominant error are related to h, not to ϵ. Also looking at the diagonal values of the Table 6.1 we see that the numerical errors are in agreement with the theoretical convergent rates of the Theorems 5.1 and 5.2.

Table 6.1

$\|u_\epsilon^* - u_\epsilon^h\|_0$ error, $h_f = 1/2048$

$\epsilon \downarrow \quad h \rightarrow$	1/8	1/16	1/32	1/64
1/16	2.7085e-04	7.7993e-05		
1/32	2.6300e-04	6.6246e-05	1.7773e-05	
1/64	2.5388e-04	5.8069e-05	1.6020e-05	1.2137e-05

$|u_\epsilon^* - u_\epsilon^h|_{1,h}$ error, $h_f = 1/2048$

1/16	0.0097	0.0067		
1/32	0.0086	0.0051	0.0036	
1/64	0.0086	0.0044	0.0025	0.0018

The Table 6.2 shows the level of approximation when each term of $u_\epsilon = u_0^h + \epsilon u_1^h + \epsilon\bar{\phi}^h + \epsilon\tilde{\phi}_\epsilon^h$ is taken into account separately. We can see that the numerical boundary layer treatment $\bar{\phi}^h$ is very effective on the L_2-norm rather than on the broken H^1-semi-norm. The improvement on the L^2 norm is an evidence that we were able to obtain, through the proper calculation of χ^*, the L^2 asymptotic behavior of the boundary

TABLE 6.2

$\epsilon = 1/64$, $h = 1/32$, $h_f = 1/1024$

	$\|\cdot\|_0$	$\|\cdot\|_{1,h}$
$u_\epsilon^* - u_0^h$	0.0287	0.0215
$u_\epsilon^* - u_0^h - \epsilon u_1^h$	0.0213	0.0026
$u_\epsilon^* - u_0^h - \epsilon u_1^h - \epsilon\bar{\phi}^h$	5.0450e-05	0.0026
$u_\epsilon^* - u_0^h - \epsilon u_1^h - \epsilon(\bar{\phi}^h + \tilde{\phi}_\epsilon^h)$	5.1865e-05	0.0025

corrector θ_ϵ in the interior of the domain Ω. We also see from the Table 6.2 that the term $\tilde{\phi}_\epsilon$ ($\bar{\phi}^h$ and $\tilde{\phi}_\epsilon$) forces the final approximation u_ϵ^h to satisfy the zero Dirichlet boundary condition without deteriorating the L^2-norm (broken H^1-semi-norm).

The second test of experiments, reported in Table 6.3, consider the discontinuous coefficient $a(y_1, y_2)$ case

$$a(y) = \begin{cases} 2 & \text{if } 2/5 < y_1 < 3/5 \text{ or } 2/5 < y_2 < 3/5 \\ 1 & \text{otherwise.} \end{cases} \quad \text{and } f = -1,$$

and we obtain the same behavior as above.

TABLE 6.3

$\|u_\epsilon^* - u_\epsilon^h\|_0$ error, $h_f = 1/2000$

$\epsilon \downarrow \quad h \rightarrow$	1/10	1/20	1/40
1/20	4.8318e-04	1.3043e-04	
1/40	4.7578e-04	1.1954e-04	3.0805e-05
1/64	2.5388e-04	5.9446e-05	1.4414e-05

$|u_\epsilon^* - u_\epsilon^h|_{1,h}$ error, $h_f = 1/2000$

$\epsilon \downarrow \quad h \rightarrow$	1/10	1/20	1/40
1/20	0.0180	0.0092	
1/40	0.0179	0.0090	0.0046
1/64	0.0086	0.0045	0.0026

Although the convergence analysis presented here is not intended for the quasi periodic case $a_{ij}(x, x/\epsilon)$, the numerical approximation presented here can be generalized for this case. This would be done by approximating the matrix $a(x, x/\epsilon)$ by $\sum_{K_j \in \mathcal{T}_h(\Omega)} a_j(x/\epsilon) I_{K_j}(x)$, where $a_j(x/\epsilon) = \frac{1}{|K_j|}\int_{z \in K_j} a_j(z, x/\epsilon)dz$, and I_{K_j} is the characteristic function of K_j. In such case, it is required to solve a cell problem for each element K_j.

7. Conclusions. We perform the convergence analysis for the proposed numerical method for approximating the solution of the equation (1.1). The error estimates obtained in the numerical experiments agree with the theoretical error estimates from the Theorems 5.1 and 5.2. The numerical method presented here is strongly based on the periodicity of the coefficients a_{ij} and on the construction of computable boundary correctors. For this reason the method has relative low computational cost with optimal error convergence rates.

We generalize results in the literature for estimating the error between u_ϵ and its first and second order asymptotic expansion under weak assumptions on the regularity of the solutions of the cell problems and the homogenized equation. We also develop and analyze constructive boundary correctors. Such analysis, permit us to develop finite element error estimates with very weak assumptions on the regularity of $a(y)$, including composite materials applications.

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