Partition of Unity Coarse Spaces and Schwarz Methods with Harmonic Overlap

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Abstract. Coarse spaces play a crucial role in making Schwarz type domain decomposition methods scalable with respect to the number of subdomains. In this paper, we consider coarse spaces based on a class of partition of unity (PU) for some domain decomposition methods including the classical overlapping Schwarz method, and the new Schwarz method with harmonic overlap. PU has been used as a very powerful tool in the theoretical analysis of Schwarz type domain decomposition methods and meshless discretization schemes. In this paper, we show that PU can also be used effectively in the numerical construction of coarse spaces. PU based coarse spaces are easy to construct and need less communication than the standard finite-element-basis-function-based coarse space in distributed memory parallel implementations. We prove the new result that the condition number of the algorithms grows only linearly with respect to the relative size of the overlap. We also introduce the additive Schwarz method (AS) with harmonic overlap (ASHO), where all functions are made harmonic in part of the overlapping regions. As a result, the communication cost and condition number of ASHO is smaller than that of AS. Numerical experiments and a conditioning theory are presented in the paper.

1 Introduction

Fast domain decomposition algorithms for elliptic problems are typically two-level methods. In this paper, we introduce and analyze two-level overlapping Schwarz methods for unstructured meshes. These methods are based on partition of unity (PU) coarse spaces and/or on the concept of harmonic overlap introduced in [4]. Work on two-level methods on unstructured meshes is not new. Several different approaches have been introduced and some can be found in [1,2,6–14,16] and papers cited therein. Related works to ours, based on two-level agglomeration techniques, can be found in [1,10]. Their analysis use a class of partition of unity coarse space based on agglomeration smoothing techniques and they have proven an upper bound for the condition number which depends quadratically on the relative overlap. In this paper, we consider coarse basis functions based on the kind of partition of unity used in the theoretical analysis of Schwarz methods. These coarse basis functions have been around for some time and can be found in some of the numerical experiments of [10]. This kind of partition of unity has a controlled decaying property on the overlapping region. In the analysis, we instead of using two distinct partition of unities, one for designing the algorithm and another one for analyzing the algorithm, as used by [10], we here use the same partition of unity for both except near the boundary. To accomplish that, we use a key argument based on a combination of Neumann-Neumann [12,9] and small overlap techniques [8] to establish a new result in which an upper bound for the condition number of the preconditioners depends only linearly on the relative overlap. We remark that this key argument for the analysis was directly inspired by the the analysis developed in RASHO [3,4].

We note that partition of unity functions do not vanish on the boundary of the original domain. Hence, they cannot be used straightforwardly as coarse basis functions since they should satisfy zero Dirichlet boundary conditions for Dirichlet boundary problems. We have tested two approaches. For the first approach, we do not include in the coarse spaces, the coarse basis functions that touch $\partial\Omega$; i.e. the boundary coarse functions. For the second approach, we modify the boundary coarse functions so that they have a controlled decaying to zero near $\partial\Omega$ and include them in the coarse spaces. We show that coarse spaces with boundary coarse functions are much more effective than without them. We first concentrate ourselves in this paper toward analyzing the second approach case. The analysis for coarse spaces without the boundary coarse basis functions will then follow easily. We refer to [11] for a modified proof of the first approach and for discussions on the choice of coarse basis functions and related rate of convergence.

The preconditioners to be discussed below are applicable for general symmetric positive definite problems. They are algebraic in the sense that the notions of subdomains, harmonic overlaps, the classification of the nodal points, etc., can all be defined in terms of the graph of the sparse matrix. In order to provide a complete and simple mathematical analysis, we restrict our discussion to a finite element problem, the Poisson problem with zero Dirichlet boundary condition. Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v), \quad \forall\ v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx \quad \text{for} \quad f \in L^2(\Omega).$$

For simplicity, let Ω be a bounded polygonal region in $\Re^2$ with a diameter of size $O(1)$. The extension of the results to $\Re^3$ can be carried out easily. Let $\mathcal{T}^h(\Omega)$ be a shape regular, quasi-uniform triangulation, of size $O(h)$, of Ω and $\mathcal{V} \subset H_0^1(\Omega)$ the finite element space consisting of continuous piecewise linear functions associated with the triangulation. The extension of the theory for the case of local quasi-uniform triangulation is also straightforward. We are interested in solving the following discrete problem associated with (1): Find $u^* \in \mathcal{V}$ such that

$$a(u^*,v) = f(v), \quad \forall\ v \in \mathcal{V}. \tag{2}$$

Using the standard basis functions, (2) can be rewritten as a linear system of equations

$$Au^* = f. \tag{3}$$

For simplicity, we understand u^* and f both as functions and vectors depending on the situation. Throughout this paper, C and C_0, are positive generic constants that are independent of any of the mesh parameters and the number of subdomains. All the domains and subdomains are assumed to be open; i.e., boundaries are not included in their definitions.

2 Notations

Given the domain Ω and triangulation $\mathcal{T}^h(\Omega)$, we assume that a domain partition has been applied and resulted in N non-overlapping subdomains $\Omega_i, i = 1, \ldots N$ of size $O(H)$, such that

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for} \quad j \neq i.$$

We define the overlapping subdomains Ω_i^δ as follows. Let Ω_i^1 be the one-overlap element extension of Ω_i, where $\Omega_i^1 \supset \Omega_i$ is obtained by including all the immediate neighboring elements $\tau_h \in \mathcal{T}^h(\Omega)$ of Ω_i such that $\overline{\tau}_h \cap \overline{\Omega}_i \neq \emptyset$. Using the idea recursively, we can define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i \subset \Omega_i^1 \subset \cdots \Omega_i^\delta.$$

Here the integer $\delta \geq 1$ indicates the level of element extension and δh is the approximate length of the extension. We note that this extension can be coded easily through the knowledge of the adjacent matrix associated to the mesh.

In this paper we consider the case of Dirichlet boundary condition on the whole $\partial\Omega$. To introduce coarse basis functions of low energy near $\partial\Omega$ we introduce a Dirichlet boundary treatment. Let Ω_B^1 be one layer of elements near the Dirichlet boundary $\partial\Omega$ and then define recursively,

$$\Omega_B^1 \subset \Omega_B^2 \cdots \Omega_B^\delta$$

with δ levels of extension by adding recursively neighboring elements. We note that we can choose δ extensions to obtain Ω_i^δ and $\hat{\delta}$ to obtain $\Omega_B^{\hat{\delta}}$. If $O(\delta) \leq \hat{\delta} \leq O(H/h)$, we obtain similar upper bounds in the analysis. Numerically, $\hat{\delta} = \delta$ gives good results.

To define and analyze the new preconditioners, we subdivide Ω_i^δ as follow. Let $\gamma_i^\delta = \partial\Omega_i^\delta \backslash \partial\Omega, i = 1, \cdots, N$; i.e., the part of the boundary of Ω_i^δ that does not belong to the physical boundary of Ω. Let $\gamma_B^\delta = \partial\Omega_B^\delta \backslash \partial\Omega$. We define the interface overlapping boundary Γ^δ as the union of all the γ_i^δ and γ_B^δ; i.e., $\Gamma^\delta = \cup_{i=B,1}^N \gamma_i^\delta$. We also need the following subsets of Ω_i^δ

- $\Gamma_i^\delta = \Gamma^\delta \cap \Omega_i^\delta$ (local interface)
- $\mathcal{N}_i^\delta = \Omega_i^\delta \backslash (\cup_{j \neq i} \Omega_j^\delta \cup \Omega_B^\delta \cup \Gamma_i^\delta)$ (non-overlapping region)
- $\mathcal{O}_i^\delta = \Omega_i^\delta \backslash (\mathcal{N}_i^\delta \cup \Gamma_i^\delta)$ (overlapping region)

We note that $\Omega_i^\delta = \mathcal{N}_i^\delta \cup \Gamma_i^\delta \cup \mathcal{O}_i^\delta$. The region $\mathcal{O}_i^\delta$ is the overlapping region of Ω_i^δ excluding the Γ_i^δ. The region $\mathcal{N}_i^\delta$ is the subregion of Ω_i^δ which does not overlap any neighboring extended subdomain $\overline{\Omega}_j^\delta$ and $\overline{\Omega}_B^\delta$. We recall that the regions $\mathcal{O}_i^\delta$ and $\mathcal{N}_i^\delta$ are open sets.

3 Overlapping Additive Schwarz (AS) Methods

We next review the AS method and then introduce a coarse space based on a partition of unity.

3.1 The AS Method without a Coarse Space

We introduce the space $\mathcal{V}_i^\delta = \mathcal{V} \cap H_0^1(\Omega_i^\delta)$ extended by zero to $\Omega \backslash \Omega_i^\delta$. It is easy to verify that

$$\mathcal{V} = \mathcal{V}_1^\delta + \mathcal{V}_2^\delta + \cdots + \mathcal{V}_N^\delta.$$

This decomposition is used in defining the classical one-level additive Schwarz algorithm [15]. Note that this decomposition is not a direct sum. Let us define $P_i^\delta : \mathcal{V} \rightarrow \mathcal{V}_i^\delta$ by: for any $u \in \mathcal{V}$

$$a(P_i^\delta u, v) = a(u, v), \quad \forall v \in \mathcal{V}_i^\delta. \tag{4}$$

Then, the classical one-level additive Schwarz operator has the form

$$P^\delta = P_1^\delta + \cdots + P_N^\delta.$$

We next introduce a partition of unity (PU) coarse space $\mathcal{V}_0^\delta$ to the AS method.

3.2 A PU Coarse Space for the AS Method

We next construct a partition of unity θ_i^δ such that $\theta_i^\delta \in \mathcal{V}_i^\delta$, $0 \leq \theta_i^\delta(x) \leq 1$, $|\nabla \theta_i^\delta(x)| \leq C/(\delta h)$, and $\sum_{i=B,1}^N \theta_i^\delta \equiv 1$. We next give one possible construction of the θ_i^δ. We first construct the function $\hat{\theta}_B^\delta \in \mathcal{V}_i^\delta$ as follows. We let $\hat{\theta}_B^\delta(x) = 1$ and $\hat{\theta}_B^\delta(x) = 0$ for nodes x on $\partial\Omega$ and $\Omega \backslash \Omega_B^\delta$, respectively. For the first layer of neighboring nodes x of $\partial\Omega$ we let $\hat{\theta}_B^\delta(x) = (\delta - 1)/\delta$, For the second layer of neighboring nodes x of $\partial\Omega$ we let $\hat{\theta}_B^\delta(x) = (\delta - 2)/\delta$, and recursively, we let $\hat{\theta}_B^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\partial\Omega$. Similarly, for $i = 1, \cdots, N$, we let $\hat{\theta}_i^\delta(x) = 1$ and $\hat{\theta}_i^\delta(x) = 0$ for nodes

x of $\overline{\Omega}_i \backslash \Omega_B^\delta$ and $\overline{\Omega} \backslash \Omega_i^\delta$, respectively. For the first layer of neighboring nodes x of $\overline{\Omega}_i \backslash \Omega_B^\delta$ we let $\hat{\theta}_i^\delta(x) = (\delta - 1)/\delta$, and recursively, we let $\hat{\theta}_i^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\overline{\Omega}_i \backslash \Omega_B^\delta$. It is easy to verify that $0 \le \hat{\theta}_i^\delta(x) \le 1$, and for quasi-uniform triangulation $|\nabla \hat{\theta}_i^\delta(x)| \le C/(\delta h)$. The partition of unity θ_i^δ is defined as

$$\theta_i^\delta = I_h\Big(\frac{\hat{\theta}_i^\delta}{\sum_{j=B,1}^N \hat{\theta}_j^\delta}\Big).$$

It is easy to verify that $\sum_{i=B,1}^N \theta_i^\delta(x) = 1, \quad 0 \le \theta_i^\delta(x) \le 1$, and $|\nabla \theta_i^\delta(x)| \le C/(\delta h), \forall x \in \bar{\Omega}$.

The PU coarse space $\mathcal{V}_0^\delta$ is given as the linear combination of the coarse basis functions $\theta_i^\delta, i = 1, \cdots, N$. Note that we do not include the θ_B^δ to the PU coarse space $\mathcal{V}_0^\delta$. Let us define $P_0 : \mathcal{V} \to \mathcal{V}_0^\delta$ by: for any $u \in \mathcal{V}$

$$a(P_0^\delta u, v) = a(u, v), \quad \forall v \in \mathcal{V}_0^\delta.$$

Then, the two-level additive Schwarz operator with the PU coarse problem P_0^δ has the form

$$P_C^\delta = \sum_{i=0}^N P_i^\delta.$$

4 AS Methods with Harmonic Overlap (ASHO)

We next introduce two ASHO methods: One without coarse space and a second one with a coarse space based on a partition of unity.

4.1 ASHO Method without a Coarse Space

We define $\widetilde{\mathcal{V}}_i^\delta$ as a subspace of $\mathcal{V}_i^\delta$ consisting of functions that are discrete harmonic at all nodes interior to $\mathcal{O}_i^\delta$, i.e. $u \in \widetilde{\mathcal{V}}_i^\delta$, if for all nodes $x_k \in \mathcal{O}_i^\delta$,

$$a(u, \phi_{x_k}) = 0.$$

Here, $\phi_{x_k} \in \mathcal{V}$ is the regular finite element basis function associated with node x_k, i.e. $\phi_{x_k}(x_k) = 1$, and $\phi_{x_k}(x_j) = 0, j \neq k$.

We define $\widetilde{\mathcal{V}}^\delta$ as a subspace of $\mathcal{V}$ defined as

$$\widetilde{\mathcal{V}}^\delta = \widetilde{\mathcal{V}}_1^\delta + \widetilde{\mathcal{V}}_2^\delta + \cdots + \widetilde{\mathcal{V}}_N^\delta.$$

We note that the above sum is not a direct sum and $\widetilde{\mathcal{V}}_i^\delta \neq \mathcal{V}_i^\delta$. We define $\widetilde{P}_i^\delta : \widetilde{\mathcal{V}}^\delta \to \widetilde{\mathcal{V}}_i^\delta$ to be the projection operators such that, for any $u \in \widetilde{\mathcal{V}}^\delta$

$$a(\widetilde{P}_i^\delta u, v) = a(u, v), \quad \forall v \in \widetilde{\mathcal{V}}_i^\delta.$$

Then, the one-level additive overlapping Schwarz method (ASHO) with harmonic overlap is defined as

$$\widetilde{P}^{\delta} = \sum_{i=1}^{N} \widetilde{P}_i^{\delta}.$$

We next introduce a PU coarse space $\widetilde{\mathcal{V}}_0^{\delta}$ for the ASHO method.

4.2 A PU Coarse Space for ASHO Method

To define the PU coarse space $\widetilde{\mathcal{V}}_0^{\delta} \subset \widetilde{\mathcal{V}}^{\delta}$, we simply modify our basis functions θ_i^{δ} to $\widetilde{\theta}_i^{\delta}$. The $\widetilde{\theta}_i^{\delta}$ is defined to be equal to θ_i^{δ} except on $\mathcal{O}_i^{\delta}$. On $\mathcal{O}_i^{\delta}$ we make $\widetilde{\theta}_i^{\delta}$ discrete harmonic.

The PU coarse space $\widetilde{\mathcal{V}}_0^{\delta}$ is given as the linear combination of the coarse basis functions $\tilde{\theta}_i^{\delta}, i = 1, \cdots, N$. We introduce $\widetilde{P}_0 : \widetilde{\mathcal{V}}^{\delta} \rightarrow \widetilde{\mathcal{V}}_0^{\delta}$ as the operator such that, for any $u \in \widetilde{\mathcal{V}}^{\delta}$,

$$a(\widetilde{P}_0^{\delta} u, v) = a(u, v), \quad \forall v \in \widetilde{\mathcal{V}}_0^{\delta} \tag{5}$$

Then, the two-level ASHO with the PU coarse problem $\widetilde{P}_0^{\delta}$ is defined as

$$\widetilde{P}_C^{\delta} = \sum_{i=0}^{N} \widetilde{P}_i^{\delta}. \tag{6}$$

5 Hybrid Methods with PU Coarse Spaces

In this paper we also consider the special hybrid Schwarz operator (see [15,13]) with the error propagation operator given by

$$(I - \sum_{i=1}^{N} P_i^{\delta})(I - P_0^{\delta}),$$

or after an additional coarse solve,

$$(I - P_0^{\delta})(I - \sum_{i=1}^{N} P_i^{\delta})(I - P_0^{\delta}). \tag{7}$$

This is a symmetric operator with which we can work essentially without any extra cost, since when forming powers of the operator (7), we can use the fact that $I - P_0^{\delta}$ is a projection, and therefore $(I - P_0^{\delta})^2 = I - P_0^{\delta}$. Subtracting the operator (7) from the identity operator I, we obtain the operator

$$P_{hyb}^{\delta} = P_0^{\delta} + (I - P_0^{\delta})(\sum_{i=1}^{N} P_i^{\delta})(I - P_0^{\delta}).$$

We also consider the harmonic overlap version for the hybrid algorithm defined by

$$\widetilde{P}^{\delta}_{hyb} = \widetilde{P}^{\delta}_{0} + (I - \widetilde{P}^{\delta}_{0})(\sum_{i=1}^{N} \widetilde{P}^{\delta}_{i})(I - \widetilde{P}^{\delta}_{0}).$$

6 Remarks about ASHO Methods

We next show that the explicit elimination of the variables associated with the overlapping nodes is not needed in order to apply $\widetilde{P}^{\delta}$ to any given vector $v \in \widetilde{\mathcal{V}}^{\delta}$.

Lemma 1. *For any $u \in \widetilde{\mathcal{V}}^{\delta}$, we have*

$$\widetilde{P}^{\delta}_{i} u = P^{\delta}_{i} u, \quad i = 1, \cdots, N.$$

Hence, $\widetilde{P}^{\delta} u = P^{\delta} u$, $u \in \widetilde{\mathcal{V}}^{\delta}$.

Proof. If $u \in \widetilde{\mathcal{V}}^{\delta}$ then

$$a(P^{\delta}_{i} u, \phi_{x_k}) = a(u, \phi_{x_k}) = 0, \quad \forall x_k \in \mathcal{O}^{\delta}_{i}.$$

Hence, $P^{\delta}_{i} u \in \widetilde{\mathcal{V}}^{\delta}_{i}$. Here, $\phi_{x_k} \in \mathcal{V}^{\delta}_{i}$ are the regular basis functions associated to the nodes x_k. To complete the proof of the lemma, we just need to verify that

$$a(P^{\delta}_{i} u, v) = a(u, v), \quad \forall v \in \widetilde{\mathcal{V}}^{\delta}_{i}. \tag{8}$$

To verify (8), we use the definition of P^{δ}_{i} (4) and that $\widetilde{\mathcal{V}}^{\delta}_{i}$ is a subset of $\mathcal{V}^{\delta}$.

We note that the solution u^* of (3) is not in the subspace $\widetilde{\mathcal{V}}^{\delta}$, therefore, the operators $\widetilde{P}^{\delta}, \widetilde{P}^{\delta}_{C}$, and $\widetilde{P}^{\delta}_{hyb}$ cannot be used to solve the linear system (3) directly. We will need to modify the right-hand side of the system (3). A reformulated (3) will be presented in Lemma 2 below. Using the matrix notations, the next lemma shows how to modify the system (3) so that its solution belongs to $\widetilde{\mathcal{V}}^{\delta}$. Let $\mathcal{O}^{\delta} = \cup_i \mathcal{O}^{\delta}_{i}$. Let $W^{\delta}_{\mathcal{O}}$ be the set of nodes associated to the degree of freedom of $\mathcal{V}^{\delta}$ in $\mathcal{O}^{\delta}$. We define the restriction operator, or a matrix, $R_{\mathcal{O}^{\delta}}\colon W \to W$ as follows

$$\left(R_{\mathcal{O}^{\delta}} v\right)(x_k) = \begin{cases} v_k & \text{if } x_k \in W_{\mathcal{O}^{\delta}} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix representation of $R^{\delta}_{\mathcal{O}}$ is given by a diagonal matrix with 1 for nodal points in the interior of $\mathcal{O}^{\delta}$ and zero for the remaining nodal points. Using this restriction operator, we define the subdomain stiffness matrix as

$$A_{\mathcal{O}^{\delta}} = R_{\mathcal{O}^{\delta}}\ A\ R^{T}_{\mathcal{O}^{\delta}},$$

which can also be obtained by the discretization of the original finite element problem on $\mathcal{O}^\delta$ with zero Dirichlet data on $\partial\mathcal{O}^\delta$ and extended by zero outside of $\mathcal{O}^\delta$. We remark that $\mathcal{O}$ is a disconnected region where $\partial\mathcal{O} = \Gamma_i^\delta \cup \partial\Omega$. Therefore, $A_{\mathcal{O}^\delta} w = f$ can be solved locally and inexpensively.

It is easy to see that the following lemma holds; see [4].

Lemma 2. *Let u^* and f be the exact solution and the right-hand side of (3), and*

$$w = R_{\mathcal{O}^\delta}^T A_{\mathcal{O}^\delta}^+ R_{\mathcal{O}^\delta} f. \tag{9}$$

Then $\widetilde{u}^* = u^* - w \in \widetilde{\mathcal{V}}^\delta$ and satisfies the following modified linear system of equations

$$A\widetilde{u}^* = f - Aw = \widetilde{f}.$$

7 Theoretical Analysis

The algorithms presented in the previous section are applicable for general sparse, symmetric positive definite linear systems. The notions of subdomains, harmonic overlaps, the classification of the regions $\mathcal{O}_i^\delta$ and $\mathcal{N}_i^\delta$ and the interfaces Γ_i^δ, etc., can all be defined in terms of the graph of the sparse matrix. In this section we provide a sharp and nearly optimal estimate for a Poisson equation discretized with a piecewise linear finite element method. We estimate the condition numbers of the operators P^δ, $\widetilde{P}^\delta$, P_C^δ, $\widetilde{P}_C^\delta$, P_{hyb}^δ, and $\widetilde{P}_{hyb}^\delta$ in terms of the fine mesh size h, the subdomain size H, and the overlapping factor δ. We shall follow the abstract additive Schwarz theory [15] to analyze the additive versions, where three assumptions have to be checked and three paramenters C_0, ω and $\rho(\mathcal{E})$ estimated. Two assumptions are trivial to check: $\omega = 1$ since we use exact solvers, and $\rho(\mathcal{E}) \le C$ since we use two-level algorithms. So our focus in the rest of the paper is in bounding C_0 for each of the preconditioned operators P^δ, P_C^δ, $\widetilde{P}_C^\delta$, and $\widetilde{P}^\delta$. To analyze the hybrid algorithms we use a result due to Mandel (Lemma 3.2 [13]) which in our context is given by

Lemma 3.

$$\lambda_{min}(P_{hyb}^\delta) \ge \lambda_{min}(P_C^\delta), \qquad \lambda_{min}(\widetilde{P}_{hyb}^\delta) \ge \lambda_{min}(\widetilde{P}_C^\delta)$$

$$\lambda_{max}(P_{hyb}^\delta) \le \lambda_{max}(P^\delta), \quad \textit{and} \quad \lambda_{max}(\widetilde{P}_{hyb}^\delta) \le \lambda_{max}(\widetilde{P}^\delta).$$

7.1 Additive Schwarz with a PU Coarse Space

Let $\mathcal{V}_{i,0}^\delta, i = 1, \cdots, N$ be the one-dimensional spaces generated by the θ_i^δ coarse basis functions. We introduce the interpolation-like operator $I_0^\delta = \sum_{i=1}^N I_{i,0}^\delta$, where the $I_{i,0}^\delta : \mathcal{V} \to \mathcal{V}_{i,0}^\delta$ is defined as follows:

$$(I_{i,0}^\delta u)(x) = \bar{u}_i^\delta \theta_i^\delta(x),$$

where

$$\bar{u}_i^\delta = \frac{\int_{\Omega_i^\delta} u dx}{\int_{\Omega_i^\delta} 1 dx},$$

is the average of u on the extended region Ω_i^δ. Here $|\Omega_i^\delta|$ is the area of the region Ω_i^δ.

We next we give a direct proof of the H_1-stability of I_0^δ. We remark that the H_1-stability is not new and an alternative proof can be found in [10].

Lemma 4. *For any $u \in \mathcal{V}$ we have*

$$a(I_0^\delta u, I_0^\delta u) \le C \frac{H}{\delta h} a(u,u). \tag{10}$$

Proof. Let c be an arbitrary constant. Using Cauchy-Schwarz inequality and the fact $|\Omega_i^\delta| = O(H^2)$ we have

$$|\bar{u}_i^\delta - c| = \frac{1}{|\Omega_i^\delta|} \left| \int_{\Omega_i^\delta} (u-c) dx \right| \le C \frac{1}{H} \|u - c\|_{L^2(\Omega_i^\delta)}. \tag{11}$$

Let Ω_j^δ be a neighbor subdomain of Ω_i^δ; i.e., $\Omega_j^\delta \cap \Omega_i^\delta \neq \emptyset$. Using a triangular inequality and the estimate (11), we obtain

$$|\bar{u}_i^\delta - \bar{u}_j^\delta| \le |\bar{u}_i^\delta - c| + |\bar{u}_j^\delta - c| \le C \frac{1}{H} \|u - c\|_{L^2(\Omega_i^\delta \cup \Omega_j^\delta)}.$$

Applying a Bramble-Hilbert argument, we have

$$\begin{aligned} |\bar{u}_i^\delta - \bar{u}_j^\delta| &\le \tfrac{C}{H} \inf_c \|u - c\|_{L^2(\Omega_i^\delta \cup \Omega_j^\delta)} \\ &\le \tfrac{C}{H} \inf_c \|u - c\|_{L^2(\Omega_i^{\delta,ext})} \\ &\le C\, |u|_{H^1(\Omega_i^{\delta,ext})}. \end{aligned} \tag{12}$$

Here $\Omega_i^{\delta,ext}$ is the union of Ω_i^δ and all of its neighbors Ω_k^δ. We note that we use here that $\Omega_i^{\delta,ext}$ has size $O(H)$ and has nice shape in order to obtain the last inequality through Friedrichs inequality. For the case when Ω_i^δ is distant $O(H)$ from the boundary $\partial\Omega$, we use another Friedrichs inequality to obtain

$$|\bar{u}_i^\delta| \le C \frac{1}{H} \|u\|_{L^2(\Omega_i^\delta)} \le C \frac{1}{H} \|u\|_{L^2(\Omega_i^{\delta,ext})} \le C |u|_{H^1(\Omega_i^{\delta,ext})}, \tag{13}$$

where $\Omega_i^{\delta,ext}$ is the union of Ω_i^δ and all of its neighbors Ω_k^δ and possibly a few more $\Omega_{k'}^\delta$ so that $\Omega_i^{\delta,ext}$ has size $O(H)$ and intersects the boundary $\partial\Omega$ on a set of measure of $O(H)$.

The rest of the proof is devoted to bound $|I_0^\delta u|^2_{H^1(\mathcal{N}_i^\delta)}$ and $|I_0^\delta u|^2_{H^1(\mathcal{O}_i^\delta)}$. For $x \in \mathcal{N}_i^\delta$, we have $\theta_j^\delta(x) = 0$, $j \neq i$, and $\theta_i^\delta(x) = 1$. Therefore, we have

$$|I_0^\delta u|^2_{H^1(\mathcal{N}_i^\delta)} = |\bar{u}_i^\delta|^2_{H^1(\mathcal{N}_i^\delta)} = 0.$$

For the region $\mathcal{O}_i^\delta$ we have

$$|u_0|^2_{H^1(\mathcal{O}_i^\delta)} = |I_0^\delta u|^2_{H^1(\mathcal{O}_i^\delta)} = |\sum_{j=B,1}^{N} \bar{u}_j^\delta \theta_j^\delta|^2_{H^1(\mathcal{O}_i^\delta)},$$

where here we have introduced, artificially, the constant $\bar{u}_B^\delta = 0$.

We next use that

$$\sum_{j=B,1}^{N} \theta_j^\delta(x) = \theta_i^\delta(x) + \sum_{j\neq i} \theta_j^\delta(x) = 1, \quad \forall x \in \bar{\Omega}$$

to obtain

$$|I_0^\delta u|^2_{H^1(\mathcal{O}_i^\delta)} = \left| \sum_{j\neq i} (\bar{u}_j^\delta - \bar{u}_i^\delta)\theta_j^\delta \right|^2_{H^1(\mathcal{O}_i^\delta)}.$$

By a triangular inequality we have

$$|I_0^\delta u|^2_{H^1(\mathcal{O}_i^\delta)} \le C \sum_{j\neq i} |(\bar{u}_j^\delta - \bar{u}_i^\delta)\theta_j^\delta|^2_{H^1(\mathcal{O}_i^\delta)} \le C \sum_{j\neq i} |\bar{u}_j^\delta - \bar{u}_i^\delta|^2 |\theta_j^\delta|^2_{H^1(\mathcal{O}_i^\delta)}.$$

We next use that $|\nabla \theta_k^\delta(x)| \le C/(\delta h)$, the fact the area of $\mathcal{O}_i^\delta$ is at most of $O(H\delta h)$, and the estimates (12) (case $j \neq B$) and (13) (case $j = B$), to obtain

$$|\bar{u}_j^\delta - \bar{u}_i^\delta|^2 |\theta_j^\delta|^2_{H^1(\mathcal{O}_i^\delta)} \le C \frac{H}{\delta h} |u|^2_{H^1(\Omega_i^{\delta,ext})}. \tag{14}$$

The estimate (10) follows by summing the contributions of all extended domains $\Omega_i^{\delta,ext}$ and using a coloring argument.

For the case when the boundary coarse basis functions θ_i^δ are not included in the coarse space, we just set $\bar{u}_i^\delta = 0$ and the proof above also holds. In this case, we use (12) for subdomains that are neighbors of boundary domains. We note that the last constant C in (13) grows when the size of $\Omega_i^{\delta,ext}$ gets larger, or equivalently, when Ω_i^δ gets more distant from $\partial\Omega$. That explain why coarse spaces without the boundary coarse basis functions are less effective than with them.

We next prove the main result of the paper.

Theorem 1. *There exists a constant $C > 0$, independent of h, δ, and H, such that*

$$\kappa(P^\delta) \le C \frac{1}{H^2}(1 + \frac{H}{\delta h}) \tag{15}$$

and

$$\kappa(P_{hyb}^{\delta}) \leq \kappa(P_C^{\delta}) \leq C(1 + \frac{H}{\delta h}). \tag{16}$$

Proof. The bound (15) is well known and it is proved in Dryja and Widlund [8]. The first inequality of (16) follows directly from the Lemma 3. What remains to complete the proof is to derive a bound for C_0; i.e., to find C_0 such that for any given $u \in \mathcal{V}$, there exist $u_i \in \mathcal{V}_i^{\delta}$, such that

$$u = \sum_{i=0}^{N} u_i, \tag{17}$$

and

$$\sum_{i=0}^{N} a(u_i, u_i) \leq C_0 a(u, u). \tag{18}$$

We define the decomposition $u = \sum_{i=0}^{N} u_i$ as follows. Let $u_0 \in \mathcal{V}_0^{\delta}$ be defined as

$$u_0 = I_0^{\delta} u = \sum_{i=1}^{N} \bar{u}_i^{\delta} \theta_i^{\delta},$$

and let $u_i \in \mathcal{V}_i^{\delta}$ be defined as

$$u_i = I_h(\vartheta_i^{\delta} u) - \bar{u}_i^{\delta} \theta_i^{\delta}, i = 1, \ldots, N.$$

Here, I_h is the standard pointwise interpolator. The piecewise linear functions $\vartheta_i^{\delta} \in H^1(\Omega_i^{\delta})$ are defined below and form a partition of unity $\sum_{i=1}^{N} \vartheta_i \equiv 1$ on $\bar{\Omega}$. It is easy to see (17) holds.

We next modify the coarse basis functions θ_i^{δ} on $(\bar{\Omega}_i^{\delta} \cap \bar{\Omega}_B^{\delta})$ to define the partition of unity ϑ_i^{δ}. We first construct the function $\hat{\vartheta}_i^{\delta} \in H^1(\Omega_i^{\delta})$. Let $\hat{\vartheta}_i^{\delta}(x) = 1$ and $\hat{\vartheta}_i^{\delta}(x) = 0$ for nodes x of $\overline{\Omega}_i$ and $\overline{\Omega} \backslash \Omega_i^{\delta}$, respectively. For the first layer of neighboring nodes x of $\overline{\Omega}_i$ we let $\hat{\vartheta}_i^{\delta}(x) = (\delta - 1)/\delta$, and recursively, we let $\hat{\vartheta}_i^{\delta}(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\overline{\Omega}_i$. The partition of unity ϑ_i^{δ} is defined as

$$\vartheta_i^{\delta} = I_h(\frac{\hat{\vartheta}_i^{\delta}}{\sum_{j=1}^{N} \hat{\vartheta}_j^{\delta}}).$$

It is easy to verify that $\sum_{i=1}^{N} \vartheta_i^{\delta}(x) = 1, \quad 0 \leq \vartheta_i^{\delta}(x) \leq 1$, and $|\nabla \vartheta_i^{\delta}(x)| \leq C/(\delta h)$, when $x \in \bar{\Omega}$, and also that $\vartheta_i^{\delta}(x) = \theta_i^{\delta}(x), i = 1, \cdots, N$, when $x \in \Omega \backslash \Omega_B^{\delta}$.

We decompose u_i as $u_i = u_i^0 + u_i^B$ where

$$u_i^0 = I_h(\theta_i^{\delta}(u - \bar{u}_i^{\delta})), \quad \text{and} \quad u_i^B = I_h((\vartheta_i^{\delta} - \theta_i^{\delta})u)), \quad i = 1, \cdots, N. \tag{19}$$

The next step is to bound $\sum_{i=0}^{N} a(u_i, u_i)$. In order to bound $a(u_0, u_0)$ we use Lemma 4. The remaining of the proof is to bound $a(u_i^0, u_i^0)$ and $a(u_i^B, u_i^B)$ since

$$a(u_i, u_i) \leq 2a(u_i^0, u_i^0) + 2a(u_i^B, u_i^B), i = 1, \cdots, N.$$

We denote $w_i = u - \bar{u}_i^\delta$. Estimating $a(u_i^0, u_i^0)$ as in the standard additive Schwarz method [8], we obtain

$$a(u_i^0, u_i^0) = |I_h(\theta_i^\delta w_i)|^2_{H^1(\Omega_i^\delta)} \leq C \left(|w_i|^2_{H^1(\Omega_i^\delta)} + \frac{1}{(\delta h)^2} \|w_i\|_{L^2(\mathcal{O}_i^\delta)} \right).$$

Using Lemma 3 in Dryja and Widlund [8]; i.e.,

$$\frac{1}{(\delta h)^2} \|w_i\|^2_{L^2(\mathcal{O}_i^\delta)} \leq C \left((1 + \frac{H}{\delta h}) |w_i|^2_{H^1(\Omega_i^\delta)} + \frac{1}{H(\delta h)} \|w_i\|^2_{L^2(\Omega_i^\delta)} \right),$$

we obtain

$$|u_i^0|^2_{H^1(\Omega_i^\delta)} \leq C \left((1 + \frac{H}{\delta h}) |w_i|^2_{H^1(\Omega_i^\delta)} + \frac{1}{H(\delta h)} \|w_i\|^2_{L^2(\Omega_i^\delta)} \right).$$

Combining these estimates and that $|w_i|^2_{H^1(\Omega_i^\delta)} = |u|^2_{H^1(\Omega_i^\delta)}$ and a Friedrichs inequality

$$\|w_i\|^2_{L^2(\Omega_i^\delta)} \leq CH^2 |u|^2_{H^1(\Omega_i^\delta)},$$

we obtain

$$a(u_i^0, u_i^0) \leq C(1 + \frac{H}{\delta h}) |u|^2_{H^1(\Omega_i^\delta)}.$$

We next bound $a(u_i^B, u_i^B)$. We note that for $i = 1, \cdots, N$

$$\vartheta_i^\delta(x) = \theta_i^\delta(x), \forall x \in \Omega \backslash \bar{\Omega}_B^\delta,$$

and therefore the support of u_i^B is $\bar{\Omega}_B^\delta \cap \bar{\Omega}_i^\delta$.

Using standard additive Schwarz methods arguments, we obtain

$$a(u_i^B, u_i^B) = |u_i^B|^2_{H^1(\Omega_i^\delta \cap \Omega_B^\delta)} \leq C \left(|u|^2_{H^1(\Omega_i^\delta \cap \Omega_B^\delta)} + \frac{1}{(\delta h)^2} \|u\|^2_{L^2(\Omega_i^\delta \cap \Omega_B^\delta)} \right).$$

Using a Friedrichs inequality for the case when the width of $\Omega_i^\delta \cap \Omega_B^\delta$ is $O(\delta h)$ and u vanishes on the larger side (u vanishes on $\partial \Omega$); i.e.,

$$\|u\|^2_{L^2(\Omega_i^\delta \cap \Omega_B^\delta)} \leq C(\delta h)^2 |u|^2_{H^1(\Omega_i^\delta \cap \Omega_B^\delta)},$$

we obtain

$$a(u_i^B, u_i^B) \leq C |u|^2_{H^1(\Omega_i^\delta \cap \Omega_B^\delta)}.$$

And the proof is complete by summing all the contributions and using a coloring argument.

The proof also holds for the case when the boundary coarse basis functions θ_i^δ are not included in the coarse space. In this case, we set $\bar{u}_i^\delta = 0$, and use that

$$\|w_i\|^2_{L^2(\Omega_i^\delta)} \leq CH^2 |u|^2_{H^1(\Omega_i^{\delta, ext})}.$$

7.2 Additive Schwarz with Harmonic Overlap Methods

We next find upper bounds for the condition numbers of the preconditioners with harmonic overlap.

Theorem 2.

$$\kappa(\widetilde{P}^{\delta}) \le C\frac{1}{H^2}(1+\frac{H}{\delta h}) \tag{20}$$

and

$$\kappa(\widetilde{P}^{\delta}_{hyb}) \le \kappa(\widetilde{P}^{\delta}_{C}) \le C(1+\frac{H}{\delta h}). \tag{21}$$

Proof. The first inequality of (21) follows directly from the Lemma 3. We next show the second inequality of (21). Let $u \in \widetilde{\mathcal{V}}^{\delta}$, and let the u_i be the decomposition (17) introduced in Theorem 1. We next define $\tilde{u}_i$ equals to u_i on $(\cup_{j=1}^{N}\mathcal{N}_j^{\delta}) \cup \Gamma_i^{\delta} \cup \partial\Omega$, and discrete harmonic on $\mathcal{O} = \cup_{j=1}^{N}\mathcal{O}_j^{\delta}$. We recall that $\mathcal{O}$ is a disconnected region and therefore discrete harmonic extension on $\mathcal{O}$ can be obtained locally. Using that $u \in \widetilde{\mathcal{V}}^{\delta}$, it is easy to see that

$$u = \sum_{i=0}^{N} \tilde{u}_i, \quad \text{and the} \quad u_i \in \widetilde{\mathcal{V}}_i^{\delta}.$$

Since the discrete harmonic extensions have the minimal semi-energy norm, we have

$$a(\tilde{u}_i,\tilde{u}_i) \le a(u_i,u_i), \quad i = 0, \cdots, N,$$

and using the bound (18) in Theorem 1 we obtain

$$\sum_{i=0}^{N} a(\tilde{u}_i,\tilde{u}_i) \le \sum_{i=0}^{N} a(u_i,u_i) \le C(1+\frac{H}{\delta h})a(u,u).$$

The proof of (20) follows the same lines as above; i.e., we first introduce a decomposition using the subspaces $\mathcal{V}_i^{\delta}$ (see [8]), and then we modify this decomposition on $\mathcal{O}$ to obtain a decomposition with functions on $\widetilde{\mathcal{V}}_i^{\delta}$.

8 Numerical Experiments and Final Remarks

In this section, we present some numerical results for solving the Poisson's equation on the unit square with zero Dirichlet boundary conditions. We compare the performance of the preconditioned Conjugate Gradient methods. As preconditioners, we consider the ASHO and AS with PU coarse spaces (given by $\widetilde{P}_C$ and P_C) and without coarse spaces (given by $\widetilde{P}$ and P), HybridHO and Hybrid with PU coarse spaces (given by $\widetilde{P}_{hyb}$ and P_{hyb}), and the AS and Hybrid with the standard non-nested coarse space; see [2,6]. In the numerical experiments below, we pay particular attention to the dependence on the number of subdomains, mesh size, and the size of overlap.

We first discuss a few implementation issues related to the preconditioner based on harmonic overlap. In order to apply the ASHO/CG and HybridHO/CG methods, it is necessary to force the solution to belong to $\widetilde{\mathcal{V}}^{\delta}$. To do so, a pre-CG-computation is needed, and it is done through the formula (9). We note, by Lemma 2, that $u = u^* - w \in \widetilde{\mathcal{V}}^{\delta}$. Hence, we can apply the regular PCG to the $\widetilde{P}^{\delta}$, $\widetilde{P}^{\delta}_{C}$, and $\widetilde{P}^{\delta}_{Hyb}$ preconditioned systems.

The exact solution of the equation is $u(x,y) = e^{5(x+y)} \sin(\pi x) \sin(\pi y)$. All subdomain problems are solved exactly. The stopping condition for CG is to reduce the initial residual by a factor of 10^{-6}. The iteration counts (iter), condition numbers (cond), maximum (max) and minimum (min) eigenvalues of the preconditioned matrix are summarized in Tables 1, 2, 3, 4, and 5.

¿From all the Tables below, it is clear that ASHO/CG (HybridHO/CG) is always better than the classical AS/CG (Hybrid/CG) in terms of the condition numbers, while they have similar behavior in terms of iteration numbers. This is an advantage because the AS based preconditioners can be modified to ASHO ones with a large saving in communications on a parallel computer with distributed memory. From Tables 1, 2, 6, and 7, it is clear that the hybrid versions with PU coarse spaces are always much better than the additive versions with PU coarse spaces. This is an important result since the extra computational cost of the hybrid ones over the additive ones is the calculation of one residual per iteration. ¿From Tables 1 and 6 we can see the effective of the PU coarse spaces. It is clear that the partition of unity coarse space makes the algorithms scalable with respect to the number of subdomains. The hybrid formulations attains the asymptotic behavior of scalability faster. From Table 3 we have the AS/CG and ASHO/CG without a coarse space. We can see a dramatic grow of iterations and condition numbers and also the ASHO is slightly better than AS preconditioner. From Tables 2 and 7 we can see that the condition number of the preconditioners all grow linearly with the size of the overlap. This is an important result since we can recover the same linear behavior as obtained for regular coarse spaces on structured meshes or non-nested coarse spaces. This results agree with the theory developed in this paper. ¿From Tables 4 and 5 we compare the PU coarse spaces with standard non-nested coarse spaces. It is clear that the hybrid versions using the partition of unity coarse spaces can perform the same or better than non-nested coarse spaces. This is an advantage since the partition of unity is easy to implement for unstructured meshes and on parallel computers. a coarse space. Finally we see, by comparing Tables 1 and 6, and by comparing with Tables 2 and 7, that all the algorithms perform considerably better if the boundary coarse functions are included in the coarse space.

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Table 1. Two-level HybridHO/CG (Hybrid/CG) and ASHO/CG (AS/CG) using PU coarse spaces for solving the Poisson's equation on a $16 * DOM \times 16 * DOM$ mesh decomposed into $DOM \times DOM$ subdomains with overlapping size $\delta = 2$.

HybridHO/CG (Hybrid/CG)

$DOM \times DOM$	iter	cond	max	min
2×2	13 (13)	5.89 (9.71)	2.19 (4.00)	0.372 (0.412)
4×4	17 (18)	6.32 (11.4)	2.20 (4.00)	0.347 (0.345)
8×8	18 (19)	6.48 (11.8)	2.20 (4.00)	0.340 (0.340)
16×16	17 (19)	6.54 (11.9)	2.20 (4.00)	0.337 (0.340)

ASHO/CG (AS/CG)

$DOM \times DOM$	iter	cond	max	min
2×2	14 (15)	7.71 (11.2)	2.52 (4.00)	0.326 (0.356)
4×4	24 (24)	12.7 (16.6)	2.79 (4.00)	0.278 (0.241)
8×8	30 (31)	16.3 (22.0)	2.85 (4.00)	0.175 (0.182)
16×16	32 (34)	17.5 (24.0)	2.85 (4.00)	0.164 (0.166)

Table 2. Two-level HybridHO/CG (Hybrid/CG) and ASHO/CG (AS/CG) using PU coarse spaces for solving the Poisson's equation on a 256×256 mesh decomposed into 16×16 subdomains with different overlapping sizes δ.

HybridHO/CG (Hybrid/CG)

δ	iter	cond	max	min
1	25 (26)	12.7 (23.5)	2.17 (4.00)	0.170 (0.170)
2	17 (19)	6.54 (11.9)	2.20 (4.00)	0.337 (0.340)
3	15 (16)	4.53 (8.07)	2.24 (4.00)	0.495 (0.495)
4	13 (14)	3.48 (6.19)	2.26 (4.00)	0.649 (0.646)

ASHO/CG (AS/CG)

δ	iter	cond	max	min
1	47 (48)	36.5 (49.7)	2.94 (4.00)	0.081 (0.081)
2	32 (34)	17.5 (24.0)	2.85 (4.00)	0.164 (0.166)
3	25 (26)	11.0 (15.4)	2.76 (4.00)	0.251 (0.260)
4	21 (22)	7.67 (11.0)	2.68 (4.00)	0.351 (0.363)

Table 3. One-level ASHO/CG (AS/CG) without coarse spaces for solving the Poisson's equation on a $16{*}DOM \times 16{*}DOM$ mesh decomposed into $DOM \times DOM$ subdomains with overlapping sizes $\delta = 2$.

ASHO/CG (AS/CG)				
$DOM \times DOM$	iter	cond	max	min
2×2	13 (14)	9.05 (16.4)	2.21 (4.00)	0.2455 (0.2445)
4×4	25 (27)	28.7 (51.8)	2.22 (4.00)	0.0772 (0.0772)
8×8	46 (48)	108. (195.)	2.22 (4.00)	0.0205 (0.0205)
16×16	89 (93)	426. (768.)	2.22 (4.00)	0.0052 (0.0052)

Table 4. Two-level Hybrid/CG (AS/CG) using non-nested coarse spaces for solving the Poisson's equation on a $16 * DOM \times 16 * DOM$ mesh decomposed into $DOM \times DOM$ subdomains with overlapping size $\delta = 2$.

Hybrid/CG (AS/CG)				
$DOM \times DOM$	iter	cond	max	min
2×2	13 (14)	7.18 (7.26)	4.04 (4.00)	0.557 (0.557)
4×4	18 (18)	7.32 (7.53)	3.98 (4.10)	0.554 (0.544)
8×8	18 (18)	7.35 (7.62)	3.98 (4.12)	0.543 (0.541)
16×16	18 (19)	7.40 (7.65)	3.99 (4.11)	0.540 (0.538)

Table 5. Two-level Hybrid/CG (AS/CG) using non-nested coarse spaces for solving the Poisson's equation on a 256×256 mesh decomposed into 16×16 subdomains with different overlapping sizes δ.

Hybrid/CG (AS/CG)				
δ	iter	cond	max	min
1	23 (23)	12.9 (13.0)	4.00 (4.02)	0.310 (0.310)
2	18 (19)	7.40 (7.65)	4.00 (4.11)	0.540 (0.538)
3	16 (17)	5.65 (6.06)	4.00 (4.24)	0.708 (0.700)
4	15 (16)	4.83 (5.41)	4.00 (4.38)	0.825 (0.809)

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Table 6. Two-level HybridHO/CG (Hybrid/CG) and ASHO/CG (AS/CG) using the PU coarse space without the boundary coarse basis functions. We solve the Poisson's equation on a $16{*}DOM \times 16{*}DOM$ mesh decomposed into $DOM \times DOM$ subdomains with overlapping size $\delta = 2$.

Hybrid/CG (AS/CG)

$DOM \times DOM$	iter	cond	max	min
2×2	13 (14)	9.06 (16.4)	2.21 (4.00)	0.245 (0.245)
4×4	22 (23)	13.7 (24.7)	2.21 (4.00)	0.162 (0.162)
8×8	27 (29)	14.9 (26.9)	2.21 (4.00)	0.149 (0.149)
16×16	29 (30)	15.3 (27.6)	2.21 (4.00)	0.145 (0.145)

ASHO/CG (AS/CG)

$DOM \times DOM$	iter	cond	max	min
2×2	13 (14)	9.05 (16.4)	2.21 (4.00)	0.245(0.245)
4×4	26 (27)	22.5 (32.6)	2.76 (4.00)	0.123 (0.123)
8×8	37 (38)	28.1 (39.5)	2.84(4.00)	0.101 (0.101)
16×16	41 (42)	29.6 (41.3)	2.86 (4.00)	0.097 (0.097)

Table 7. Two-level HybridHO/CG (Hybrid/CG) and ASHO/CG (AS/CG) using the PU coarse space without boundary coarse basis functions. We solve the Poisson's equation on a 256×256 mesh decomposed into 16×16 subdomains with different overlapping sizes δ.

HybridHO/CG (Hybrid/CG)

δ	iter	cond	max	min
1	41 (43)	30.5(56.3)	2.17 (4.00)	0.071 (0.071)
2	29 (30)	13.7 (27.6)	2.21 (4.00)	0.162 (0.145)
3	24 (26)	10.2 (18.1)	2.25 (4.00)	0.220 (0.220)
4	21 (21)	7.62 (13.4)	2.27 (4.00)	0.298 (0.298)

ASHO/CG (AS/CG)

δ	iter	cond	max	min
1	59 (60)	62.5 (84.8)	2.94 (4.00)	0.047 (0.047)
2	41 (42)	29.6 (41.3)	2.86 (4.00)	0.097 (0.097)
3	32 (34)	18.6 (26.9)	2.77 (4.00)	0.149 (0.149)
4	28 (29)	13.2 (19.6)	2.70 (4.00)	0.204 (0.204)

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