

# FETI and BDD Preconditioners for Stokes-Mortar-Darcy Systems

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## Abstract

We consider the coupling across an interface of a fluid flow and a porous media flow. The differential equations involve Stokes equations in the fluid region and Darcy equations in the porous region, and a coupling through an interface with Beaver-Joseph-Saffman transmission conditions. The discretization consists of  $P2/P1$  triangular Taylor-Hood finite elements in the fluid region, the lowest order triangular Raviart-Thomas finite elements in the porous region, and the mortar piecewise constant Lagrange multipliers on the interface. We allow for nonmatching meshes across the interface. Due to the small values of the permeability parameter  $\kappa$  of the porous medium, the resulting discrete symmetric saddle point system is very ill conditioned. We design and analyze preconditioners based on the Finite Element by Tearing and Interconnecting (FETI) and Balancing Domain Decomposition (BDD) methods and derive a condition number estimate of order  $C_1(1 + \frac{1}{\kappa})$  for the preconditioned operator. In case the fluid discretization is finer than the porous side discretization, we derive a better estimate of order  $C_2(\frac{\kappa+1}{\kappa+(h^p)^2})$  for the FETI preconditioner. Here  $h^p$  is the mesh size of the porous side triangulation. The constants  $C_1$  and  $C_2$  are independent of the permeability  $\kappa$ , the fluid viscosity  $\nu$ , and the mesh ratio across the interface. Numerical experiments confirm the sharpness of the theoretical estimates.

**Keywords.** Stokes-Darcy coupling, mortar, balancing domain decomposition, FETI, saddle point problems, nonmatching grids, discontinuous coefficients, mortar elements

**AMS subject classifications.** 65N55, 65N30, 65N22, 35Q30, 35Q35, 76D07

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## 1 Introduction

We consider the coupling across an interface of a fluid flow and a porous media flow. The model consists of Stokes equations in the fluid region, Darcy equations for the filtration velocity in the porous medium, and an adequate transmission condition for coupling of these equations through an interface. Such problem appears in several applications such as well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. There are works that address numerical analysis issues of this model. For inf-sup conditions and approximation results associated to the continuous and discrete formulations for Stokes-Laplacian systems we refer [14, 15], for Stokes-Darcy systems we refer [29, 37, 2], for Stokes-Mortar-Darcy systems we refer [39, 25], and for DG discretizations [11, 39]. For studies on preconditioning analysis for Stokes-Laplacian systems we refer [12, 13, 16, 17], and for Stokes-Darcy systems we refer [3]. In this paper, we are interested in Balancing Domain Decomposition (BDD) and Finite Element by Tearing and Interconnecting (FETI) preconditioned Conjugate Gradient methods for *Stokes-Mortar-Darcy* systems. For general references on BDD methods we mention [21, 33, 34, 38, 40] and for FETI methods [21, 23, 28, 35, 40] for FETI methods; see also [18, 22, 30, 31, 32, 41, 42].

In this paper we both extend some preliminary results contained in [24] and introduce and analyze new methods. We note that the BDD-I preconditioner introduced in [24] is not effective for small permeabilities (in real applications permeabilities are very small) while the preconditioner BDD-II in [24] requires constructing interface base functions which are orthogonal in the Stokes inner product (this construction is very expensive and impractical because it requires, as a precomputational step, solving many Stokes problems). Here in this paper we circumvent these issues by introducing a dual formulation and considering a FETI based methods. We propose and analyze Finite Element by Tearing and Interconnecting (FETI) methods and present numerical experiments in order to verify the theory. We note that the analysis of the FETI algorithms for Stokes-Mortar-Darcy problems is very challenging due to the following issues: 1) The mortar map from the Stokes to the Darcy side has a large kernel since the Stokes velocity space is in general richer than the Darcy velocity space on the interface; 2) The trace space of the Stokes velocity ( $H^{1/2}$ ) is more regular than the trace space of the Darcy flux ( $H^{-1/2}$ ), and due to a priori error estimates, see [29, 39, 25], the Stokes side must be chosen as the master side; 3) The energy associated to the Darcy region is much larger than the energy associated to the Stokes region due to the small value of the permeability. Such issues imply that the master side must be chosen on the Stokes side and where the energy is smaller and velocity space is richer. The mathematical analysis under this choice is very hard to analyze even for simpler problems such as for transmission problems with discontinuous coefficients using Mortar or DG discretizations [21, 19, 20]. For problems where both the smallest coefficient and the finest mesh are placed on the master side, as far as we know, there are no optimal preconditioners developed in the literature for transmission problems, and typically there is a

condition to rule out such choice.

The rest of the paper is organized as follows: in Section 2 we present the Stokes/Darcy coupling model. In Section 3 we describe the weak formulation of this model. In Section 4 we introduce a finite element discretization. In Section 5 we study the primal and dual formulation of the discrete problem. Section 6 is dedicated to present a complete analysis of the BDD-I preconditioner introduced in [24]. In Section 7, we design and analyze the FETI preconditioner; see Lemma 3 and Theorem 4. In particular we obtain the condition number estimate of order  $C_1(1 + \frac{1}{\kappa})$  for this preconditioner and also prove Theorem 7 which gives a better estimate of order  $C_2(\frac{\kappa+1}{\kappa+(h^p)^2})$  for the FETI preconditioner in case the fluid discretization is finer than the porous side discretization; the case where the Stokes mesh is not a refinement of the Darcy mesh is also discussed; see Remark 8. In Section 7 we also consider more general fluid bilinear forms by allowing the presence of a tangential interface fluid velocity energy, see Remark 10, and also translate the FETI results to analyze certain BDD methods; see Remark 9.

Here  $h^p$  is the mesh size of the porous side triangulation. The constants  $C_1$  and  $C_2$  are independent of the permeability  $\kappa$ , the fluid viscosity  $\nu$ , and the mesh ratio across the interface. In Section 8 we present numerical results that confirm the theoretical estimates concerning the BDD and the FETI preconditioners.

## 2 Problem Setting

Let  $\Omega^f, \Omega^p \subset \mathbb{R}^n$  be polyhedral subdomains, define  $\Omega := \text{int}(\overline{\Omega^f} \cup \overline{\Omega^p})$  and  $\Gamma := \partial\Omega^f \cap \partial\Omega^p$ , with outward unit normal vectors  $\boldsymbol{\eta}^i$  on  $\partial\Omega^i, i = f, p$ . The tangent vectors on  $\Gamma$  are denoted by  $\boldsymbol{\tau}_1$  ( $n = 2$ ), or  $\boldsymbol{\tau}_l, l = 1, 2$  ( $n = 3$ ). The exterior boundaries are  $\Gamma^i := \partial\Omega^i \setminus \Gamma, i = f, p$ . Fluid velocities are denoted by  $\mathbf{u}^i : \Omega^i \rightarrow \mathbb{R}^n, i = f, p$ , and pressures by  $p^i : \Omega^i \rightarrow \mathbb{R}, i = f, p$ .

We consider Stokes equations in the fluid region  $\Omega^f$  and Darcy equations for the filtration velocity in the porous medium  $\Omega^p$ . More precisely, we have the following systems of equations in each subdomain:

$$\begin{array}{cc} \text{Stokes equations} & \text{Darcy equations} \\ \left\{ \begin{array}{l} -\nabla \cdot T(\mathbf{u}^f, p^f) = \mathbf{f}^f \quad \text{in } \Omega^f \\ \nabla \cdot \mathbf{u}^f = g^f \quad \text{in } \Omega^f \\ \mathbf{u}^f = \mathbf{h}^f \quad \text{on } \Gamma^f \end{array} \right. & \left\{ \begin{array}{l} \mathbf{u}^p = -\frac{\kappa}{\nu} \nabla p^p \quad \text{in } \Omega^p \\ \nabla \cdot \mathbf{u}^p = g^p \quad \text{in } \Omega^p \\ \mathbf{u}^p \cdot \boldsymbol{\eta}^p = h^p \quad \text{on } \Gamma^p. \end{array} \right. \quad (1) \end{array}$$

Here  $T(\mathbf{v}, p) := -pI + 2\nu \mathbf{D}\mathbf{v}$ , where  $\nu$  is the fluid viscosity,  $\mathbf{D}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$  is the linearized strain tensor and  $\kappa$  denotes the rock permeability. For simplicity on the analysis, we assume that  $\kappa$  is a real positive constant. We impose the following conditions:

1. *Interface matching conditions across  $\Gamma$* ; see [14, 15, 16, 29] and references therein.

- (a) *Conservation of mass across  $\Gamma$* :  $\mathbf{u}^f \cdot \boldsymbol{\eta}^f + \mathbf{u}^p \cdot \boldsymbol{\eta}^p = 0$  on  $\Gamma$ .
- (b) *Balance of normal forces across  $\Gamma$* :  $p^f - 2\nu\boldsymbol{\eta}^{fT}\mathbf{D}(\mathbf{u}^f)\boldsymbol{\eta}^f = p^p$  on  $\Gamma$ .
- (c) *Beavers-Joseph-Saffman condition*: This condition is a kind of empirical law that gives an expression for the component of the Cauchy stress tensor in the tangential direction of  $\Gamma$ ; see [4] and [27]. It is expressed by:

$$\mathbf{u}^f \cdot \boldsymbol{\tau}_l = -\frac{\sqrt{k}}{\alpha^f} 2\boldsymbol{\eta}^{fT}\mathbf{D}(\mathbf{u}^f)\boldsymbol{\tau}_l \quad l = 1, n-1; \text{ on } \Gamma.$$

2. *Compatibility condition*: The divergence and boundary data satisfy (see [25]),

$$\langle g^f, 1 \rangle_{\Omega^f} + \langle g^p, 1 \rangle_{\Omega^p} - \langle \mathbf{h}^f \cdot \boldsymbol{\eta}^f, 1 \rangle_{\Gamma^f} - \langle h^p, 1 \rangle_{\Gamma^p} = 0.$$

### 3 Weak Formulation

In this section we present the weak version of the coupled system of partial differential equations introduced above. Without loss of generality, we consider  $\mathbf{h}^f = \mathbf{0}$ ,  $g^f = 0$ ,  $h^p = 0$  and  $g^p = 0$  in (1); see [25].

The problem can be formulated as: *Find  $(\mathbf{u}, p, \lambda) \in \mathbf{X} \times M_0 \times \Lambda$  such that for all  $(\mathbf{v}, q, \mu) \in \mathbf{X} \times M_0 \times \Lambda$ :*

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_\Gamma(\mathbf{v}, \lambda) & = f(\mathbf{v}) \\ b(\mathbf{u}, q) & = 0 \\ b_\Gamma(\mathbf{u}, \mu) & = 0, \end{cases} \quad (2)$$

where  $\mathbf{X} = \mathbf{X}^f \times \mathbf{X}^p := H_0^1(\Omega^f, \Gamma^f)^n \times \mathbf{H}_0(\text{div}, \Omega^p, \Gamma^p)$  and  $M_0$  is the subset of  $M := L^2(\Omega^f) \times L^2(\Omega^p) \equiv L^2(\Omega)$  of pressures with zero average value in  $\Omega$ . Here  $H_0^1(\Omega^f, \Gamma^f)$  denotes the subspace of  $H^1(\Omega^f)$  of functions that vanish on  $\Gamma^f$ . The space  $\mathbf{H}_0(\text{div}, \Omega^p, \Gamma^p)$  consists of functions in  $\mathbf{H}(\text{div}, \Omega^p)$  with zero normal trace on  $\Gamma^p$ , where

$$\mathbf{H}(\text{div}, \Omega^p) := \{ \mathbf{v} \in L^2(\Omega^p)^n : \text{div } \mathbf{v} \in L^2(\Omega^p) \}.$$

For the Lagrange multiplier space we consider  $\Lambda := H^{1/2}(\Gamma)$ . See [25] for a discussion on the choice of the Lagrange multipliers space  $\Lambda$  and how to derive the weak formulation (2) and other equivalent weak formulations; see also [29].

The global bilinear forms are

$$a(\mathbf{u}, \mathbf{v}) := a_{\alpha^f}^f(\mathbf{u}^f, \mathbf{v}^f) + a^p(\mathbf{u}^p, \mathbf{v}^p) \quad \text{and} \quad b(\mathbf{v}, p) := b^f(\mathbf{v}^f, p^f) + b^p(\mathbf{v}^p, p^p),$$

with local bilinear forms  $a_{\alpha^f}^f, b^f$  and  $b^p$  defined by

$$a_{\alpha^f}^f(\mathbf{u}^f, \mathbf{v}^f) := 2\nu(\mathbf{D}\mathbf{u}^f, \mathbf{D}\mathbf{v}^f)_{\Omega^f} + \sum_{\ell=1}^{n-1} \frac{\nu\alpha^f}{\sqrt{\kappa}} \langle \mathbf{u}^f \cdot \boldsymbol{\tau}_\ell, \mathbf{v}^f \cdot \boldsymbol{\tau}_\ell \rangle_\Gamma, \quad \mathbf{u}^f, \mathbf{v}^f \in \mathbf{X}^f, \quad (3)$$

$$a^p(\mathbf{u}^p, \mathbf{v}^p) := \left(\frac{\nu}{\kappa} \mathbf{u}^p, \mathbf{v}^p\right)_{\Omega^p}, \quad \mathbf{u}^p, \mathbf{v}^p \in \mathbf{X}^p, \quad (4)$$

$$b^f(\mathbf{v}^f, q^f) := -(q^f, \nabla \cdot \mathbf{v}^f)_{\Omega^f}, \quad \mathbf{v}^f \in \mathbf{X}^f, \quad q^f \in M^f, \quad (5)$$

$$b^p(\mathbf{v}^p, p^p) := -(p^p, \nabla \cdot \mathbf{v}^p)_{\Omega^p}, \quad \mathbf{v}^p \in \mathbf{X}^p, \quad p^p \in M^p, \quad (6)$$

and with weak conservation of mass bilinear form defined by

$$b_\Gamma(\mathbf{v}, \mu) := \langle \mathbf{v}^f \cdot \boldsymbol{\eta}^f, \mu \rangle_\Gamma + \langle \mathbf{v}^p \cdot \boldsymbol{\eta}^p, \mu \rangle_\Gamma, \quad \mathbf{v} = (\mathbf{v}^f, \mathbf{v}^p) \in \mathbf{X}, \quad \mu \in \Lambda. \quad (7)$$

The second duality pairing of (7) is interpreted as  $\langle \mathbf{v}^p \cdot \boldsymbol{\eta}^p, E\boldsymbol{\eta}^p(\mu) \rangle_{\partial\Omega^p}$ . Here  $E\boldsymbol{\eta}^p$  is any continuous lift-in operator from  $H^{1/2}(\Gamma)$  to  $H^{1/2}(\partial\Omega^p)$ ; recall that  $\Gamma \subset \partial\Omega^p$  and  $\mathbf{v} \in \mathbf{H}_0(\text{div}, \Omega^p, \Gamma^p)$ . It easy to see that this duality pairing is independent of the lift-in operator  $E\boldsymbol{\eta}^p$ . This duality pairing is an extension of the  $L^2(\Gamma)$  inner product when  $\mathbf{v}^p \cdot \boldsymbol{\eta}^p$  and  $\mu$  are  $L^2(\Gamma)$  functions, see [25].

The functional  $f$  in the right hand side of (2) is defined by

$$f(\mathbf{v}) := f^f(\mathbf{v}^f) + f^p(\mathbf{v}^p), \quad \text{for all } \mathbf{v} = (\mathbf{v}^f, \mathbf{v}^p) \in \mathbf{X},$$

where  $f^i(\mathbf{v}^i) := (\mathbf{f}^i, \mathbf{v}^i)_{L^2(\Omega^i)}$  for all  $\mathbf{v}^i \in \mathbf{X}^i$ ,  $i = f, p$ .

The bilinear forms  $a_{\alpha^f}^f, b^f$  are associated to Stokes equations and the bilinear forms  $a^p, b^p$  to Darcy law. The bilinear form  $a_{\alpha^f}^f$  includes interface matching conditions 1.b and 1.c above. The bilinear form  $b_\Gamma$  is used to impose the weak version of the interface matching condition 1.a above. We have the following lemma that addresses the well-posedness of the problem.

**Lemma 1** (See [25, 29]) *There exists  $\beta > 0$  such that*

$$\inf_{\substack{(q, \mu) \in M_0 \times \Lambda \\ (q, \mu) \neq 0}} \sup_{\substack{\mathbf{v} \in \mathbf{X} \\ \mathbf{v} \neq 0}} \frac{b(\mathbf{v}, q) + b_\Gamma(\mathbf{v}, \mu)}{\|\mathbf{v}\|_{\mathbf{X}} (\|q\|_M + \|\mu\|_\Lambda)} \geq \beta > 0. \quad (8)$$

where  $\|\mathbf{v}\|_{\mathbf{X}}^2 := \|\mathbf{v}^f\|_{H_0^1(\Omega_f)^2}^2 + \|\mathbf{v}^p\|_{\mathbf{H}(div, \Omega_p)}^2$ . This inf-sup condition, together with the fact that  $a_{\alpha^f}^f$  is  $\mathbf{X}^f \times \mathbf{H}(div^0, \Omega^p)$ -elliptic and  $a_{\alpha^f}^f, b$  and  $b_\Gamma$  are bounded, guarantees the well-posedness of the problem (2).

## 4 Discretization

From now on we consider only the two dimensional case. We note that the ideas developed in the following can be easily extended to case of three dimensional subdomains.

We assume that  $\Omega^i$ ,  $i = f, p$ , are *two dimensional* polygonal subdomains. Let  $\mathcal{T}_{h^i}^i(\Omega^i)$  be a geometrically conforming shape regular and quasi-uniform triangulation of  $\Omega^i$  with mesh size parameter  $h^i$ ,  $i = f, p$ . We do not assume that these two triangulations match at the interface  $\Gamma$ . For the fluid region, let  $\mathbf{X}_{h^f}^f$  and  $M_{h^f}^f$  be  $P2/P1$  triangular Taylor-Hood finite elements; see [7, 8, 10]. More precisely,

$$\mathbf{X}_{h^f}^f := \left\{ \mathbf{u} \in \mathbf{X}^f : \begin{array}{l} \forall K \in \mathcal{T}_{h^f}^f(\Omega_f), \mathbf{u}_K = \hat{\mathbf{u}}_K \circ F_K^{-1} \\ \text{and } \hat{\mathbf{u}}_K \in P_2(\hat{K})^2 \end{array} \right\} \cap C^0(\bar{\Omega}^f)^2, \quad (9)$$

where  $\mathbf{u}_K := \mathbf{u}|_K$  and

$$M_{h^f}^f := \left\{ p \in L^2(\Omega^f) : \begin{array}{l} \forall K \in \mathcal{T}_{h^f}^f(\Omega_f), p_K = \hat{p}_K \circ F_K^{-1} \\ \text{and } \hat{p}_K \in P_1(\hat{K}) \end{array} \right\} \cap C^0(\bar{\Omega}^f).$$

Denote  $\overset{\circ}{M}_{h^f}^f \subset M_{h^f}^f$  the discrete fluid pressures with zero average value in  $\Omega^f$ . For the porous region, let  $\mathbf{X}_{h^p}^p \subset \mathbf{X}^p$  and  $M_{h^p}^p \subset L^2(\Omega^p)$  be the lowest order Raviart-Thomas finite elements based on triangles; see [7, 10]. Let  $\overset{\circ}{M}_{h^p}^p \subset M_{h^p}^p$  be the subset of pressures in  $M_{h^p}^p$  with zero average value in  $\Omega^p$ .

Define  $\mathbf{X}_h := \mathbf{X}_{h^f}^f \times \mathbf{X}_{h^p}^p \subset \mathbf{X}$  and  $M_h := M_{h^f}^f \times M_{h^p}^p \subset L^2(\Omega^f) \times L^2(\Omega^p)$ . Note that in the definition of the discrete velocities we assume that the boundary conditions are included, i.e., for  $\mathbf{v}_{h^f}^f \in \mathbf{X}_{h^f}^f$  we have  $\mathbf{v}_{h^f}^f = \mathbf{0}$  on  $\Gamma^f$  and for  $\mathbf{v}_{h^p}^p \in \mathbf{X}_{h^p}^p$  we have that  $\mathbf{v}_{h^p}^p \cdot \boldsymbol{\eta}^p = 0$  on  $\Gamma^p$ .

Let  $\mathcal{T}_{h^p}^p(\Gamma)$  be the restriction to  $\Gamma$  of the porous side triangulation  $\mathcal{T}_{h^p}^p(\Omega^p)$ . For the Lagrange multipliers space we choose piecewise constant functions on  $\Gamma$  with respect to the triangulation  $\mathcal{T}_{h^p}^p(\Gamma)$ ,

$$\Lambda_{h^p} := \left\{ \lambda : \lambda|_{e_j^p} = \lambda_{e_j^p} \text{ is constant in each edge } e_j^p \text{ of } \mathcal{T}_{h^p}^p(\Gamma) \right\}, \quad (10)$$

i.e., the *master* is on the fluid region side and the *slave* is on the porous region side; see [5, 6, 21, 43]. The choice of piecewise constant Lagrange multipliers leads to a nonconforming approximation on  $\Lambda_{h^p}$  since piecewise constant functions do not belong to  $H^{1/2}(\Gamma)$ . For the analysis of this nonconforming discretization and a priori error estimates we refer to [25].

## 5 Primal and Dual Formulations

In order to simplify the notation and since there is no danger of confusion, we will denote the finite element functions and the corresponding vector represen-

tation by the same symbol, i.e., when writing finite element functions we will drop the indices  $h^i$ . Recall that we have the pair of spaces  $(\mathbf{X}_h, M_h)$  associated to the coupled problem and spaces associated to each subproblem:  $(\mathbf{X}_{h^f}^f, M_{h^f}^f)$  and  $(\mathbf{X}_{h^p}^p, M_{h^p}^p)$ . We will keep the subscript  $h^i$ ,  $i = f, p$ , in the notation for local subspaces  $\mathbf{X}_{h^f}^f, M_{h^f}^f, \mathbf{X}_{h^p}^p$  and  $M_{h^p}^p$ .

Since we are interested in preconditioning issues we assume  $\alpha^f = 0$  in the definition of the fluid side local bilinear form  $a_{\alpha^f}^f$  in (3). We denote  $a^f = a_0^f$ . See Remark 10 for the case  $\alpha^f > 0$ .

With the discretization chosen in Section 4 we obtain the following symmetric saddle point linear system

$$\left[ \begin{array}{cc|cc|c} A^f & B^{fT} & 0 & 0 & C^{fT} \\ B^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A^p & B^{pT} & -C^{pT} \\ 0 & 0 & B^p & 0 & 0 \\ \hline C^f & 0 & -C^p & 0 & 0 \end{array} \right] \begin{bmatrix} \mathbf{u}^f \\ p^f \\ \mathbf{u}^p \\ p^p \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}^f \\ g^f \\ \mathbf{f}^p \\ g^p \\ 0 \end{bmatrix} \quad (11)$$

with matrices  $A^i, B^i, C^i$  and columns vectors  $\mathbf{f}^i, g^i$ ,  $i = f, p$ , defined by

$$\begin{aligned} a^i(\mathbf{u}^i, \mathbf{v}^i) &= \mathbf{v}^{iT} A^i \mathbf{u}^i, \\ b^i(\mathbf{u}^i, q^i) &= q^{iT} B^i \mathbf{u}^i, \\ (\mathbf{u}^i \cdot \boldsymbol{\eta}^f, \mu)_\Gamma &= \mu^T C^i \mathbf{u}^i, \\ f^i(\mathbf{v}^i) &= \mathbf{v}^{iT} \mathbf{f}^i, \\ g^i(q^i) &= q^{iT} g^i. \end{aligned} \quad (12)$$

Matrix  $A^f$  corresponds to  $\nu$  times the discrete version of the linearized stress tensor on  $\Omega^f$ . Note that in the case  $\alpha^f > 0$ , the bilinear form  $a_{\alpha^f}^f$  in (3) includes a boundary term; see Remark 10. The matrix  $A^p$  corresponds to  $\nu/\kappa$  times a discrete  $L^2$ -norm on  $\Omega^p$ . Matrix  $-B^i$  is the discrete divergence in  $\Omega^i$ ,  $i = f, p$ , and matrices  $C^f$  and  $C^p$  correspond to the matrix form of the discrete conservation of mass on  $\Gamma$ . Note that  $\nu$  can be viewed as a scaling factor since it appears in both matrices  $A^f$  and  $A^p$ . Therefore, it is not relevant for preconditioning issues.

Consider the following partition of the degrees of freedom: For  $i = f, p$ , let

$$\begin{bmatrix} \mathbf{u}_I^i \\ p_I^i \\ u_\Gamma^i \\ \bar{p}^i \end{bmatrix} \begin{array}{l} \text{Interior displacements + tangential velocities on } \Gamma, \\ \text{Interior pressures with zero average in } \Omega^i, \\ \text{Interface outward } \textit{normal velocities} \text{ on } \Gamma, \\ \text{Constant pressure in } \Omega^i. \end{array}$$

Then, for  $i = f, p$ , we have the block structure:

$$A^i = \begin{bmatrix} A_{II}^i & A_{\Gamma I}^{iT} \\ A_{\Gamma I}^i & A_{\Gamma \Gamma}^i \end{bmatrix}, \quad B^i = \begin{bmatrix} B_{II}^i & B_{\Gamma I}^{iT} \\ 0 & \bar{B}^{iT} \end{bmatrix} \quad \text{and} \quad C^i = [ 0 \quad 0 \quad \tilde{C}^i \quad 0 ].$$

Note that the  $(2, 1)$  entry of  $B^i$  corresponds to integrating an *interior* velocity against a constant pressure, then it vanishes due to the divergence theorem. We have the following matrix representation of the coupled problem in (11):

$$\left[ \begin{array}{cccc|cccc|c} A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{II}^f & 0 & B_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\Gamma I}^f & B_{II}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} & 0 & 0 & 0 & 0 & \tilde{C}^{fT} \\ 0 & 0 & \bar{B}^f & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & A_{II}^p & B_{II}^{pT} & A_{\Gamma I}^{pT} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{II}^p & 0 & B_{\Gamma I}^p & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{\Gamma I}^p & B_{II}^{pT} & A_{\Gamma\Gamma}^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{B}^p & 0 & 0 \\ \hline 0 & 0 & \tilde{C}^f & 0 & 0 & 0 & -\tilde{C}^p & 0 & 0 \end{array} \right] \left[ \begin{array}{c} \mathbf{u}_I^f \\ p_I^f \\ u_\Gamma^f \\ \bar{p}^f \\ \mathbf{u}_I^p \\ p_I^p \\ u_\Gamma^p \\ \bar{p}^p \\ \lambda \end{array} \right] = \left[ \begin{array}{c} \mathbf{f}_I^f \\ g_I^f \\ f_\Gamma^f \\ \bar{g}^f \\ \mathbf{f}_I^p \\ g_I^p \\ f_\Gamma^p \\ \bar{g}^p \\ 0 \end{array} \right]. \quad (13)$$

Following [21, 38], we choose the following matrix representation in each subdomain  $\Omega^i$ ,  $i = f, p$ ,

$$\left[ \begin{array}{cc|cc} A_{II}^i & B_{II}^{iT} & A_{\Gamma I}^{iT} & 0 \\ B_{II}^i & 0 & B_{\Gamma I}^i & 0 \\ \hline A_{\Gamma I}^i & B_{II}^{iT} & A_{\Gamma\Gamma}^i & \bar{B}^{iT} \\ 0 & 0 & \bar{B}^i & 0 \end{array} \right] = \left[ \begin{array}{c|c} K_{II}^i & K_{\Gamma I}^{iT} \\ \hline K_{\Gamma I}^i & K_{\Gamma\Gamma}^i \end{array} \right]. \quad (14)$$

## 5.1 The primal formulation

From the last equation in (13) we see that the mortar condition on  $\Gamma$  (using the Darcy side as the slave side) can be imposed as  $u_\Gamma^p = (\tilde{C}^p)^{-1} \tilde{C}^f u_\Gamma^f = \Pi u_\Gamma^f$ , where  $\Pi$  is the  $L^2(\Gamma)$  projection on the space of piecewise constant functions on each subinterval  $e^p \in \mathcal{T}_{h^p}^p(\Gamma)$ . We note that  $\tilde{C}^p$  is a diagonal matrix for the lowest order Raviart-Thomas elements.

Now we eliminate  $\mathbf{u}_I^i$ ,  $p_I^i$ ,  $i = f, p$ , and  $\lambda$ , to obtain the following (saddle point) Schur complement

$$S \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} = \begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix}. \quad (15)$$

Here  $S$  is given by

$$\begin{aligned} S &:= \begin{bmatrix} S_\Gamma^f & \bar{B}^{fT} & 0 \\ \bar{B}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tilde{\Pi}^T \begin{bmatrix} S_\Gamma^p & 0 & \bar{B}^{pT} \\ 0 & 0 & 0 \\ \bar{B}^p & 0 & 0 \end{bmatrix} \tilde{\Pi} \\ &= \tilde{S}^f + \tilde{S}^p \end{aligned}$$

$$= \left[ \begin{array}{c|cc} S_\Gamma^f + \Pi^T S_\Gamma^p \Pi & \bar{B}^{fT} & \Pi^T \bar{B}^{pT} \\ \hline \bar{B}^f & 0 & 0 \\ \bar{B}^p \Pi & 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{array} \right],$$

where

$$\tilde{\Pi} := \begin{bmatrix} \Pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{B}^T := [\bar{B}^{fT} \quad \Pi^T \bar{B}^{pT}]. \quad (16)$$

Here, we have denoted

$$\tilde{S}^f := \begin{bmatrix} S_\Gamma^f & \bar{B}^{fT} & 0 \\ \bar{B}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{S}^p := \tilde{\Pi}^T \begin{bmatrix} S_\Gamma^p & 0 & \bar{B}^{pT} \\ 0 & 0 & 0 \\ \bar{B}^p & 0 & 0 \end{bmatrix} \tilde{\Pi}. \quad (17)$$

The local matrices  $S_\Gamma^i$  and  $\bar{B}^i$  and the local Schur complement  $S^i$  are given by

$$S^i = \begin{bmatrix} S_\Gamma^i & \bar{B}^{iT} \\ \bar{B}^i & 0 \end{bmatrix} := K_{\Gamma\Gamma}^i - K_{\Gamma I}^i (K_{\Gamma\Gamma}^i)^{-1} K_{\Gamma I}^{iT}, \quad i = p, f. \quad (18)$$

The right hand side of (15) is given by

$$\begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix} = \left\{ \begin{bmatrix} f_\Gamma^f \\ \bar{g}^f \\ 0 \end{bmatrix} - \begin{bmatrix} K_{\Gamma I}^f (K_{\Gamma\Gamma}^f)^{-1} \begin{bmatrix} f_I^f \\ g_I^f \end{bmatrix} \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} \Pi^T f_\Gamma^p \\ 0 \\ \bar{g}^p \end{bmatrix} - \tilde{\Pi}^T \begin{bmatrix} K_{\Gamma I}^p (K_{\Gamma\Gamma}^p)^{-1} \begin{bmatrix} f_I^p \\ g_I^p \end{bmatrix} \\ 0 \end{bmatrix} \right\}.$$

We note that the reduced system (15), as well as the original system (13), is solvable when  $\bar{b}^f + \bar{b}^p = 0$ , and the solution is unique when we restrict to pressures with zero average value on  $\Omega$ .

From now on we only work with functions defined on  $\Gamma$  and extended inside the subdomain using the discrete Stokes and Darcy problems. It is convenient to define the space

$$V_\Gamma := \left\{ v_\Gamma = (v_\Gamma^f, v_\Gamma^p) : v_\Gamma^f = \mathcal{SH}(v^f \cdot \boldsymbol{\eta}^f|_\Gamma) \text{ and } v_\Gamma^p = \mathcal{DH}(v^p \cdot \boldsymbol{\eta}^p|_\Gamma) \right\} \quad (19)$$

and

$$M_0^h := \left\{ q \in M^h : q^i = \text{piec. const. in } \Omega^i, i = f, p, \text{ and } \int_{\Omega^f} q^f + \int_{\Omega^p} q^p = 0 \right\}. \quad (20)$$

Here  $\mathcal{SH}$  ( $\mathcal{DH}$ ) is the velocity component of the discrete Stokes (Darcy) harmonic extension operator that maps discrete interface normal velocity  $u_\Gamma^f \in H_{00}^{1/2}(\Gamma)$  (respectively  $u_\Gamma^p \in (H^{1/2}(\Gamma))'$ ) to the solution of following problem:

Find  $\mathbf{u}^i \in \mathbf{X}_{h^i}^i$  and  $p^i \in M_{h^i}^i$  such that for all  $\mathbf{v}^i \in \mathbf{X}_{h^i}^i$  and  $q^i \in M_{h^i}^i$ ,  $i = f, p$ , we have:

$$\begin{cases} a^f(\mathcal{S}\mathcal{H}u^f, \mathbf{v}^f) + b^f(\mathbf{v}^f, p^f) = 0 \\ b^f(\mathcal{S}\mathcal{H}u^f, q^f) = 0 \\ \mathcal{S}\mathcal{H}u^f \cdot \boldsymbol{\eta}^f = u_\Gamma^f \quad \text{on } \Gamma \\ \mathcal{S}\mathcal{H}u^f = \mathbf{0} \quad \text{on } \Gamma^f, \end{cases} \quad (21)$$

and

$$\begin{cases} a^p(\mathcal{D}\mathcal{H}u^p, \mathbf{v}^p) + b^p(\mathbf{v}^p, p^p) = 0 \\ b^p(\mathcal{D}\mathcal{H}u^p, q^p) = 0 \\ \mathcal{D}\mathcal{H}u^p \cdot \boldsymbol{\eta}^p = u_\Gamma^p \quad \text{on } \Gamma \\ \mathcal{D}\mathcal{H}u^p \cdot \boldsymbol{\eta}^p = 0 \quad \text{on } \Gamma^p. \end{cases} \quad (22)$$

The degrees of freedom associated with  $\mathcal{S}\mathcal{H}u^f \cdot \boldsymbol{\tau}^f$  on  $\Gamma$  are free. This corresponds to imposing the natural boundary condition  $\boldsymbol{\tau}^T \mathbf{D}(\mathcal{S}\mathcal{H}u^f) \boldsymbol{\eta}_f = 0$  on  $\Gamma$  which is the the expression for interface condition of Beavers-Joseph-Saffman with  $\alpha^f = 0$ .

For  $i = f, p$ , define the normal trace component of  $\mathbf{X}_{h^i}^i$  by

$$Z_{h^i}^i = \{ \mathbf{v}^i \cdot \boldsymbol{\eta}^i|_\Gamma : \mathbf{v}^i \in \mathbf{X}_{h^i}^i \}. \quad (23)$$

Associated with the coupled problem (13) we introduce the *balanced subspace*

$$V_{\Gamma, \bar{B}} := \left\{ v_\Gamma^f \in Z_{h^f}^f : (v_\Gamma^f, \Pi v_\Gamma^f) \in V_\Gamma, \text{ and } \int_\Gamma v_\Gamma^f \cdot \boldsymbol{\eta}_f = 0 \right\}, \quad (24)$$

with  $V_\Gamma$  defined in (19); see [38]. Observe that  $V_{\Gamma, \bar{B}} = \text{Ker } \bar{B}$ , where  $\bar{B}$  is defined in (16) and (18). Then for  $v_\Gamma^f \in V_{\Gamma, \bar{B}}$  we have  $\bar{B}v_\Gamma^f = 0$ . We will refer to functions  $v_\Gamma^f \in V_{\Gamma, \bar{B}}$  as *balanced functions*. If  $v_\Gamma^p = \Pi v_\Gamma^f$  and  $v_\Gamma^f$  is a balanced function then we also say that  $v_\Gamma^p$  is a balanced function or the pair  $(v_\Gamma^f, \Pi v_\Gamma^f)$  is balanced.

## 5.2 Dual formulation

In the system (13), we first eliminate the unknowns  $\mathbf{u}_I^f, p_I^f$  and  $\mathbf{u}_I^p, p_I^p$ . We obtain

$$\left[ \begin{array}{cc|cc|c} S_\Gamma^f & \bar{B}^{fT} & 0 & 0 & \tilde{C}^{fT} \\ \bar{B}^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & S_\Gamma^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\ 0 & 0 & \bar{B}^p & 0 & 0 \\ \hline \tilde{C}^f & 0 & -\tilde{C}^p & 0 & 0 \end{array} \right] \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ u_\Gamma^p \\ \bar{p}^p \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{b}^f \\ \tilde{b}^p \\ 0 \end{bmatrix}, \quad (25)$$

where right hand side of (25) is given by

$$\begin{bmatrix} \tilde{b}^f \\ \tilde{b}^p \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} \left[ \begin{array}{c} f_\Gamma^f \\ \bar{g}^f \end{array} \right] - K_{\Gamma I}^f (K_{\Gamma\Gamma}^f)^{-1} \left[ \begin{array}{c} f_I^f \\ g_I^f \end{array} \right] \\ \left[ \begin{array}{c} f_\Gamma^p \\ \bar{g}^p \end{array} \right] - K_{\Gamma I}^p (K_{\Gamma\Gamma}^p)^{-1} \left[ \begin{array}{c} f_I^p \\ g_I^p \end{array} \right] \\ 0 \end{bmatrix}}{0}.$$

Here  $S_\Gamma^i$ ,  $K_{II}^i$  and  $K_{I\Gamma}^i$ ,  $i = f, p$ , are defined in (18) and (14).

Let  $N_i := \begin{bmatrix} \tilde{C}^i & 0 \end{bmatrix}$  and consider  $S^i$ ,  $i = f, p$ , defined in (18). Then the matrix in the left hand side of (25) can be rewritten as

$$\left[ \begin{array}{c|c|c} S^f & 0 & N^{fT} \\ \hline 0 & S^p & -N^{pT} \\ \hline N^f & -N^p & 0 \end{array} \right].$$

Now we eliminate the unknowns  $u_\Gamma^f, \bar{p}^f$  and  $u_\Gamma^p, \bar{p}^p$ . We end up with the reduced system

$$F\lambda = c, \quad (26)$$

where the operator  $F$  is defined by

$$F := N^f (S^f)^{-1} N^{fT} + N^p (S^p)^{-1} N^{pT}, \quad (27)$$

and the right hand side  $c$  is given by

$$\begin{aligned} c = N^f (S^f)^{-1} \left\{ \left[ \begin{array}{c} f_\Gamma^f \\ \bar{g}^f \end{array} \right] - K_{\Gamma I}^f (K_{\Gamma\Gamma}^f)^{-1} \left[ \begin{array}{c} f_I^f \\ g_I^f \end{array} \right] \right\} - \\ N^p (S^p)^{-1} \left\{ \left[ \begin{array}{c} f_\Gamma^p \\ \bar{g}^p \end{array} \right] - K_{\Gamma I}^p (K_{\Gamma\Gamma}^p)^{-1} \left[ \begin{array}{c} f_I^p \\ g_I^p \end{array} \right] \right\}. \end{aligned}$$

Note that  $F$  is positive semidefinite and since a discrete Lagrange multiplier in  $\Lambda_{h^p}$  does not have necessarily zero mean average value on  $\Gamma$ , then, the operator  $F$  has one simple zero eigenvalue corresponding to a constant Lagrange multiplier. The above linear system, as well as the original linear system (13), is solvable for zero mean right hand side, i.e., for  $c^T \cdot (1, \dots, 1) = 0$ .

## 6 BDD Preconditioner

In this section we design and analyze a BDD type preconditioner for the Schur complement system (15); see [9, 21, 40] and also [1, 20, 33, 38, 41]. For the sake of simplicity on the analysis we assume that  $\Gamma = \{1\} \times (0, 1)$ ,  $\Omega^f = (1, 2) \times (0, 1)$  and  $\Omega^p = (0, 1) \times (0, 1)$ . We introduce the velocity coarse space on  $\Gamma$  as the

span of the normal velocity  $v_0 = y(1 - y)$  (with  $v_0$  also denoting its vector representation). Define:

$$R_0 := \begin{bmatrix} v_0^T & 0 \\ 0 & I_{2 \times 2} \end{bmatrix}, \quad S_0 := R_0 S R_0^T \quad \text{and} \quad Q_0 := R_0^T S_0^\dagger R_0. \quad (28)$$

The system (15) is solvable when the right hand side satisfy  $\bar{b}^f + \bar{b}^p = 0$  with uniqueness of the solution in the space of vectors with pressure component having zero average value on  $\Omega$ . Then, we have that  $S_0$  is invertible restricted to vectors with pressure component in  $M_0^h$  defined in (20). The low dimensionality of the coarse space (which is spanned by  $v_0$  and a constant pressure per subdomain  $\Omega^i, i = f, p$ .) and the fact that the functions  $v_0$  is independent of the triangulation parameters imply stable discrete inf-sup condition for the coarse problem.

Denote  $\tilde{S}_0 := v_0^T S_\Gamma v_0$  and  $\tilde{S} := \bar{B} v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T$ . We can write (see (16) and (28))

$$S_0 = \begin{bmatrix} \tilde{S}_0 & (\bar{B} v_0)^T \\ \bar{B} v_0 & 0 \end{bmatrix}.$$

A simple calculation using the formula for the inverse of a saddle point matrix gives

$$Q_0 = \begin{bmatrix} v_0 \tilde{S}_0^{-1} v_0^T - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T & v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \\ \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T & \tilde{S}^{-1} \end{bmatrix},$$

and using (16) we obtain

$$Q_0 S = \begin{bmatrix} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma + v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} & 0 \\ \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - \tilde{S}^{-1} \bar{B} & I \end{bmatrix},$$

or  $Q_0 S = \begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{G} & I \end{bmatrix}$ . Here, we have defined

$$\begin{aligned} \mathcal{P} &:= \left( v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma - v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma \right) + v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T \tilde{S}^{-1} \bar{B} \\ \mathcal{G} &:= \tilde{S}^{-1} \bar{B} - \tilde{S}^{-1} \bar{B} v_0 \tilde{S}_0^{-1} v_0^T S_\Gamma. \end{aligned}$$

With this notation we have that  $I - Q_0 S = \begin{bmatrix} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix}$ . Elementary calculations show that  $\mathcal{P}^2 = \mathcal{P}$  and  $\bar{B}(I - \mathcal{P}) = 0$ , hence  $I - \mathcal{P}$  is a projection and its image is contained on the balanced subspace defined in (24); see also [38].

Given a residual  $r = \begin{bmatrix} f_\Gamma^T & \bar{g}^T \end{bmatrix}^T$ , the coarse problem  $Q_0 r$ , with  $Q_0$  defined in (28), is the solution of the coupled problem (13) with one velocity degree of freedom ( $v_0$ ), and a constant pressure per subdomain  $\Omega^i, i = f, p$ , with mean zero in  $\Omega = \text{int}(\bar{\Omega}^f \cup \bar{\Omega}^p)$ . Note that the matrix  $S_0$  defined in (28) can be

computed easily and in order to ensure zero mean pressure on  $\Omega$  we can use a Lagrange multiplier.

For balanced functions  $v_\Gamma^f$  and  $u_\Gamma^f$ , the  $S_\Gamma$ -inner product is defined by (see (16)):

$$\langle u_\Gamma^f, v_\Gamma^f \rangle_{S_\Gamma} := \langle S_\Gamma u_\Gamma^f, v_\Gamma^f \rangle = u_\Gamma^{fT} S_\Gamma v_\Gamma^f.$$

Recall that  $\bar{B}u_\Gamma^f=0$  when  $u_\Gamma^f$  is balanced. Then, on this subspace of balanced functions, the  $S_\Gamma$  inner product coincides with the  $S$ -inner product defined by

$$\left\langle \begin{bmatrix} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{bmatrix}, \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} \right\rangle_S := \begin{bmatrix} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{bmatrix}^T S \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} = \begin{bmatrix} v_\Gamma^f \\ \bar{q} \end{bmatrix}^T \begin{bmatrix} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma^f \\ \bar{p} \end{bmatrix},$$

where  $\bar{p}^T = [\bar{p}^p \quad \bar{p}^p]^T$ . Consider the BDD preconditioner operator given by

$$S_N^{-1} := Q_0 + (I - Q_0 S) (\tilde{S}^f)^\dagger (I - S Q_0), \quad (29)$$

where  $\tilde{S}^f$  is defined in (17); see [21, 38]. The notation  $(\tilde{S}^f)^\dagger$  stands for the pseudo-inverse of  $\tilde{S}^f$ , i.e.,

$$(\tilde{S}^f)^\dagger = \begin{bmatrix} (S^f)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

with  $S^f$  defined in (18). The preconditioned operator is given by

$$\begin{aligned} S_N^{-1} S &= Q_0 S + (I - Q_0 S) (\tilde{S}^f)^\dagger S (I - Q_0 S) \\ &= \begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{G} & I \end{bmatrix} + \begin{bmatrix} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix} (\tilde{S}^f)^\dagger \begin{bmatrix} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix} \begin{bmatrix} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix} \end{aligned} \quad (30)$$

Note that applying  $(S^f)^{-1}$  to a vector  $\begin{bmatrix} u_\Gamma^f \\ \bar{p} \end{bmatrix}$  is equivalent to solving the linear system

$$\begin{bmatrix} A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 \\ B_{II}^f & 0 & B_{I\Gamma}^f & 0 \\ A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} \\ 0 & 0 & \bar{B}^f & 0 \end{bmatrix} \begin{bmatrix} w_I^f \\ s_I^f \\ w_\Gamma^f \\ \bar{s}^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_\Gamma^f \\ \bar{p}^f \end{bmatrix}.$$

If  $u_\Gamma^f$  is balanced, so is balanced the velocity component of  $(S^f)^{-1} \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \end{bmatrix}$ . Then using elementary calculations with the matrices in (30) we obtain that

$$\langle S_N^{-1} S \begin{bmatrix} u_\Gamma \\ \bar{p} \end{bmatrix}, \begin{bmatrix} v_\Gamma \\ \bar{q} \end{bmatrix} \rangle_S = \langle (S_\Gamma^f)^{-1} S_\Gamma u_\Gamma, v_\Gamma \rangle_{S_\Gamma},$$

for  $u_\Gamma, v_\Gamma \in \text{Range}(I - \mathcal{P})$ . In order to bound the condition number of the preconditioned operator  $S_N^{-1}S$ , we only need to analyze the condition of the operator  $(S_\Gamma^f)^{-1}S_\Gamma$ . Note that

$$c\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \leq \langle (S^f)^{-1} S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \leq C\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma}$$

is equivalent to

$$c\langle S^f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \leq C\langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \quad (31)$$

The next theorem shows that the condition number estimate for the BDD method introduced in (29) is of order  $O(1 + \frac{1}{\kappa})$ , where  $\kappa$  is the permeability of the porous medium; see (1).

**Theorem 2** *If  $u_\Gamma^f$  is a balanced function then*

$$\langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \prec \left(1 + \frac{1}{\kappa}\right) \langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle.$$

**Proof.** The lower bound follows trivially from  $\tilde{S}_\Gamma^f$  and  $\tilde{S}_\Gamma^p$  being positive on the subspace of balanced functions. Next we concentrate on the upper bound.

Let  $v_\Gamma^f$  be a balanced function and  $v_\Gamma^p = \Pi v_\Gamma^f$ . Define  $\mathbf{v}^p = \mathcal{DH}v_\Gamma^p$ ; see (22). Using properties of the discrete operator  $\mathcal{DH}$ , see [36], we obtain

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle = a^p(\mathbf{v}^p, \mathbf{v}^p) \asymp \frac{\nu}{\kappa} \|v_\Gamma^p\|_{(H^{1/2})'(\Gamma)}^2.$$

Using the  $L_2$ -stability property of mortar projection  $\Pi$ , we have

$$\|v_\Gamma^p\|_{(H^{1/2})'(\Gamma)}^2 \prec \|v_\Gamma^p\|_{L^2(\Gamma)}^2 = \|v_\Gamma^f\|_{L^2(\Gamma)}^2 \prec \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2.$$

With  $\mathcal{SH}$  defined in (21), define  $\mathbf{v}^f = \mathcal{SH}v_\Gamma^f$ . Using properties of  $\mathcal{SH}$ , see [38], we have

$$\nu \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2 \asymp a^f(\mathbf{v}^f, \mathbf{v}^f)$$

and then

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle \prec \frac{1}{\kappa} \langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \quad (32)$$

This gives the upper bound and finishes the proof.  $\blacksquare$

Recall that we consider the preconditioned projected conjugate gradient method applied to the Schur complement problem (15). We have written the algorithm in Figure 1.

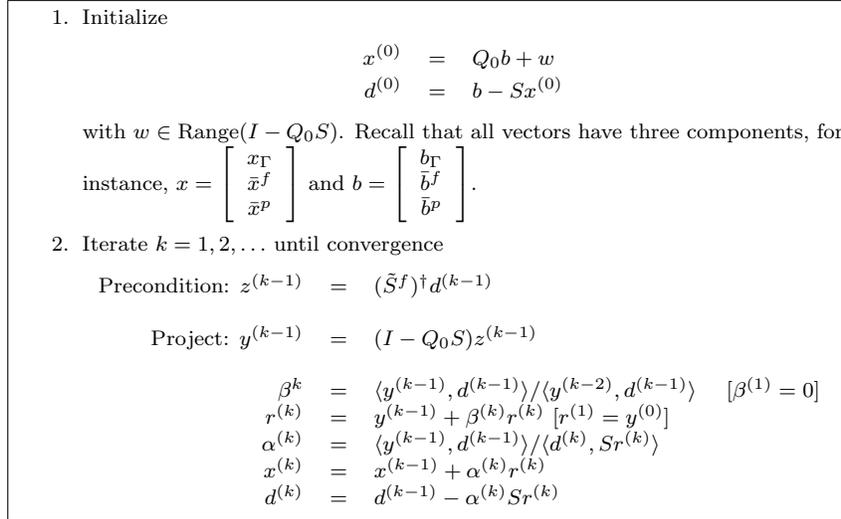


Figure 1: Implementation of the projected preconditioned conjugate gradient algorithm for the system (15) involving the BDD preconditioner (29).

## 7 FETI Preconditioner

In this section we analyze a FETI preconditioner for the reduced linear system (26); see [9, 21, 40] and also [23, 28, 35]. Recall the definition of  $F$  in (27). We propose the following preconditioner

$$(N^p)^\dagger (S^p) (N^p)^\dagger T, \tag{33}$$

where  $(N^p)^\dagger$  is the pseudo-inverse  $(N^p)^\dagger = [ \tilde{C}^p \quad 0 ]$ .

Note that after computing the action of  $(S^f)^{-1}$  and  $(S^p)^{-1}$ , in the application of  $F$  to a zero average Lagrange multiplier, we end up with balanced functions. Therefore, in order to apply the preconditioned operator  $(N^p)^\dagger (S^p) (N^p)^\dagger T F$  to a zero mean Lagrange multiplier, we do not need to solve a coarse problem at the beginning of the CG, neither inside of the CG iteration.

The FETI preconditioner in (33) can be considered as the dual preconditioner of the BDD preconditioner defined in (29); see the proof of Lemma 3 below.

Recall the definition of  $S^i$ ,  $i = f, p$ , in (18) and the definition of space of balanced functions  $V_\Gamma = V_\Gamma^f \times V_\Gamma^p$  in (24) and (23). We prove the following result.

**Lemma 3** *Let  $\lambda \in \Lambda_{hp} \cap L_0^2(\Gamma)$  be a zero mean Lagrange multiplier. Then*

$$\langle N^f (S^f)^{-1} N^{fT} \lambda, \lambda \rangle \prec \frac{1}{\kappa} \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle.$$

**Proof.** Consider a zero mean Lagrange multiplier  $\lambda$ . Define  $t = (S^p)^{-\frac{1}{2}} N^p T \lambda$  and  $w^f = N^f T \lambda$ . Then it is enough to prove that

$$\|(S^f)^{-\frac{1}{2}} w^f\|^2 \prec \|t\|^2.$$

Since  $w^f$  is balanced, i.e.,  $w^f \in V_{\Gamma}^f$ , we have that

$$\begin{aligned} \|(S^f)^{-\frac{1}{2}} w^f\|^2 &= \sup_{z^f \in Z_{h_f}^f} \frac{\langle (S^f)^{-\frac{1}{2}} w^f, z^f \rangle^2}{\|z^f\|^2} \\ &= \sup_{v^f \text{ balanced}} \frac{\langle w^f, v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\ &= \sup_{v^f \text{ balanced}} \frac{\langle \lambda, N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\ &= \sup_{v^f \text{ balanced}} \frac{\langle (S^p)^{-\frac{1}{2}} N^p \lambda, (S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2}. \end{aligned}$$

Then using the Cauchy-Schwarz inequality and (32) in the proof of Theorem 2, we have

$$\begin{aligned} \|(S^f)^{-\frac{1}{2}} w^f\|^2 &= \sup_{v^f \text{ balanced}} \frac{\langle t, (S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f \rangle^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\ &\leq \|t\|^2 \sup_{v^f \text{ balanced}} \frac{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v^f\|^2}{\|(S^f)^{\frac{1}{2}} v^f\|^2} \\ &\prec \frac{1}{\kappa} \|t\|^2. \end{aligned}$$

■

Using Lemma 3 we can derive the following estimate for the condition number of the FETI preconditioner defined in (33).

**Theorem 4** *Let  $\lambda$  be a zero mean Lagrange multiplier. Then*

$$\langle N^p (S^p)^{-1} N_p^T \lambda, \lambda \rangle \prec \langle F \lambda, \lambda \rangle \prec \left(1 + \frac{1}{\kappa}\right) \langle N^p (S^p)^{-1} N^p T \lambda, \lambda \rangle.$$

The condition number estimate  $O(\frac{\kappa+1}{\kappa})$  can be improved in the case where the fluid side triangulation is finer than the porous side triangulation. This case has some advantages when  $\kappa$  is small. In order to fix ideas and simplify notation we analyze in detail the case where the triangulation of the fluid side is a *refinement* of the porous side triangulation. In particular, in Theorem 7, we will prove that the condition of the FETI preconditioned operator is of order  $O(\frac{\kappa+1}{\kappa+(h^p)^2})$  in this simpler situation. The analysis that we will present to prove Theorem 7 can be extended easily for the case where the fluid side triangulation

is finer than (and not necessarily a refinement of) the porous side triangulation; see Remark 8.

We assume that the fluid side discretization on  $\Gamma$ ,  $\mathcal{T}_{h^f}^f(\Omega^f)|_\Gamma$ , is a refinement of the corresponding porous side discretization,  $\mathcal{T}_{h^p}^p(\Omega^p)|_\Gamma$ . That is, assume that  $h^p = rh^f$  for some positive integer  $r$ . We will refer to this assumption as the *nested refinement assumption*. For  $j = 1, \dots, m^p$ , we introduce the normal fluid velocity  $\phi_j^f$  as the  $P2$  bubble function defined on  $\mathcal{T}_{h^p}^p(\Omega^p)|_\Gamma$  and with support on the interval  $e_j^p = \{0\} \times [(j-1)h^p, jh^p]$ . Recall that we are using  $P2/P1$  Taylor-Hood discretization on the fluid side. Under the nested refinement assumption we have that  $\phi_j^f \in Z_{h^f}^f$  with  $Z_{h^f}^f$  defined in (23). Denote  $Z_{h^f,b}^f$  as the subspace of  $Z_{h^f}^f$  spanned by all  $\phi_j^f$ ,  $j = 1, \dots, m^p$ , and set  $Z_{h^f,0}^f$  as the subspace of  $Z_{h^f}^f$  spanned by functions with zero average on all edges  $e_j^p$ ,  $j = 1, \dots, m^p$ . Note that  $Z_{h^f,b}^f$  and  $Z_{h^f,0}^f$  form a direct sum for  $Z_{h^f}^f$  and the image  $\Pi Z_{h^f,0}^f$  is the zero vector.

Before deriving the condition number estimate of the FETI preconditioner under the nested refinement assumption we first prove a preliminary lemma.

**Lemma 5** *Assume that  $h^p = rh^f$ , where  $r$  is a positive integer. If  $v_{\Gamma,b}^f \in Z_{h^f,b}^f$  and  $v_{\Gamma,b}^f$  is a balanced function then*

$$\langle S_\Gamma^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \prec \frac{\kappa}{(h^p)^2} \langle S_\Gamma^p \Pi v_{\Gamma,b}^f, \Pi v_{\Gamma,b}^f \rangle.$$

**Proof.** Let  $v_{\Gamma,b}^f = \sum_{j=1}^{m^p} \beta_j \phi_j^f \in Z_{h^f,b}^f \subset Z_{h^f}^f$  and note that since the basis functions  $\phi_j^f$ ,  $j = 1, \dots, m^p$ , do not overlap each other on  $\Gamma$ , they are orthogonal in  $L^2(\Gamma)$  and also in  $H_0^1(\Gamma)$ . Then

$$\|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2 = \sum_{j=1}^{m^p} \beta_j^2 \|\phi_j^f\|_{L^2(\Gamma)}^2 \asymp h^p \sum_{j=1}^{m^p} \beta_j^2, \quad (34)$$

and

$$|v_{\Gamma,b}^f|_{H^1(\Gamma)}^2 = \sum_{j=1}^{m^p} \beta_j^2 |\phi_j^f|_{H_0^1(e_j^p)}^2 \asymp \frac{1}{h^p} \sum_{j=1}^{m^p} \beta_j^2. \quad (35)$$

Using (34), (35) and an interpolation estimate we see that

$$\|v_{\Gamma,b}^f\|_{H_0^{1/2}(\Gamma)}^2 \asymp \sum_{j=1}^{m^p} \beta_j^2 \asymp \frac{1}{h^p} \|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2.$$

Note also that

$$\langle S^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \leq a^f (\mathcal{S}H v_{\Gamma,b}^f, \mathcal{S}H v_{\Gamma,b}^f) \asymp \nu \|v_{\Gamma,b}^f\|_{H_0^{1/2}(\Gamma)}^2.$$

Denote by  $z_{\Gamma,b}^p = \sum_{j=1}^{m^p} \rho_j \chi_{e_j^p}$  the unique piecewise constant function such that  $\Pi v_{\Gamma,b}^f = z_{\Gamma,b}^p$ . Observe that  $|\rho_j| \asymp |\beta_j|$ ,  $j = 1, \dots, m^p$ . We obtain

$$\langle S_{\Gamma}^f v_{\Gamma,b}^f, v_{\Gamma,b}^f \rangle \prec \frac{\nu}{h^p} \|v_{\Gamma,b}^f\|_{L^2(\Gamma)}^2 \asymp \frac{\nu}{h^p} \|z_{\Gamma,b}^p\|_{L^2(\Gamma)}^2 \quad (36)$$

$$\prec \frac{\nu}{(h^p)^2} \|z_{\Gamma,b}^p\|_{(H^{1/2})'(\Gamma)}^2 \asymp \frac{\kappa}{(h^p)^2} \langle S_{\Gamma}^p z_{\Gamma,b}^p, z_{\Gamma,b}^p \rangle, \quad (37)$$

where we have used an inverse inequality for piecewise constant functions. ■

We now translate Lemma 5 in a result concerning to our dual preconditioner.

**Lemma 6** *Assume that  $h^p = rh^f$ , where  $r$  is a positive integer and let  $\lambda$  be a zero mean Lagrange multiplier. Then*

$$\frac{(h^p)^2}{\kappa} \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle \prec \langle N^f (S^f)^{-1} N^{fT} \lambda, \lambda \rangle.$$

**Proof.** We proceed as before. Let  $t = (S^f)^{-\frac{1}{2}} N^{fT} \lambda$  and  $w = N^p \lambda$ . Then

$$\begin{aligned} \|(S^p)^{-\frac{1}{2}} w\|^2 &= \sup_{z^f \in Z_{h^f}^f} \frac{\langle (S^p)^{-\frac{1}{2}} w, z^f \rangle^2}{\|z^p\|^2} \quad (38) \\ &= \sup_{v^p \text{ balanced}} \frac{\langle w, v^f \rangle^2}{\|(S^p)^{\frac{1}{2}} v^p\|^2} \\ &= \sup_{v^p \text{ balanced}} \frac{\langle \lambda, N^p v^p \rangle^2}{\|(S^p)^{\frac{1}{2}} v^p\|^2} \\ &= \sup_{v_b^f \text{ balanced}} \frac{\langle \lambda, N^f v_b^f \rangle^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\ &= \sup_{v_b^f \text{ balanced}} \frac{\langle (S^f)^{-\frac{1}{2}} N^{fT} \lambda, (S^f)^{\frac{1}{2}} v_b^f \rangle^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\ &\leq \|t\|^2 \sup_{v_b^f \text{ balanced}} \frac{\|(S^f)^{\frac{1}{2}} v_b^f\|^2}{\|(S^p)^{\frac{1}{2}} (N^p)^{-1} N^f v_b^f\|^2} \\ &\prec \frac{\kappa}{(h^p)^2} \|t\|^2, \end{aligned}$$

where the last step follows from Lemma 5. ■

From Lemmas 3 and 6, the next theorem follows.

**Theorem 7** *Assume that  $h^p = rh^f$ , where  $r$  is a positive integer. Let  $\lambda$  be a zero mean Lagrange multiplier, then*

$$\left(1 + \frac{(h^p)^2}{\kappa}\right) \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle \prec \langle F \lambda, \lambda \rangle \prec \left(1 + \frac{1}{\kappa}\right) \langle N^p (S^p)^{-1} N^{pT} \lambda, \lambda \rangle.$$

We solve the system (26) using preconditioned conjugate gradient. We have written the algorithm in Figure 2.

**Remark 8** *Theorem 7 can be extended for the case where  $h^f \leq 2h^p$ . We only need to extend the argument given in the proof of Lemma 5. The basic idea in the proof of Lemma 5 is to associate a bubble function  $\phi_j^f \in Z_{h^f}^f$  to each porous side element  $e_j^p$ ,  $j = 1, \dots, m^p$ , in such a way that we can construct a one to one and continuous map  $v_{\Gamma,b}^f \mapsto z_{\Gamma,b}^p$ . The bubble functions  $\phi_j^f$ ,  $j = 1, \dots, m^p$ , can be chosen orthogonal in  $L^2(\Gamma)$  and in  $H_0^1(\Gamma)$ . This can also be done when  $h^f \leq h^p$ . The smaller the  $h^f$ , the closer is the size of the support of the bubble  $\phi_j^f$  to the size of the element  $e_j^p$  since more and more elements  $e^f$  can be associated to only one element  $e^p$ . This construction can also be carried out in the case  $h^p < h^f \leq 2h^p$  where non-orthogonal Taylor-Hood basis functions must be used. This last situation leads to the appearance of an additional constant that depends on the non-orthogonality; see Section 8.*

**Remark 9** *We note that Lemma 5 can be used directly to obtain a bound for the balancing domain decomposition preconditioner similar to the one presented in Section 6 but with  $\tilde{S}^p$  instead of  $\tilde{S}^f$  in (29); see Proposition 2 of [24]. In this case an additional variable elimination is needed. We have to eliminate the component of the normal fluid velocity in the space  $Z_{h^f,0}^f$  and work with the Schur complement with respect to the space  $Z_{h^f,b}^f$ . This is rather difficult to implement (we can use Lagrange multipliers in this case). Then passing to the dual preconditioner permit us to take advantage of the case where the fluid side discretization on  $\Gamma$  is a refinement of the corresponding porous side discretization.*

**Remark 10** *Theorems 2, 4 and 7 are also valid for the case  $\alpha^f > 0$  in (3). To see this we need to compare, for different values of  $\alpha^f$ , the energy of discrete extensions for a given normal velocity defined on  $\Gamma$ . Given the outward normal velocity  $v_\Gamma^f$  on  $\Gamma$ , let  $\mathcal{SH}_{\alpha^f} v_\Gamma^f$  denote the discrete harmonic extension in the sense of  $(a_{\alpha^f}^f, b^f)$ , that is, the solution of problem (21) with  $a^f$  replaced by  $a_{\alpha^f}^f$ . Recall that  $a^f = a_0^f$ , where  $a_0^f = a_{\alpha^f}$  when  $\alpha^f = 0$ , and therefore,  $\mathcal{SH} v_\Gamma^f = \mathcal{SH}_0 v_\Gamma^f$ . Note that in (21) we have imposed the natural boundary condition  $\tau^T \mathbf{D}(\mathcal{SH} u^f) \boldsymbol{\eta}_f = 0$  on  $\Gamma$ . Now we define another extension denoted by  $\widehat{\mathcal{SH}} v_\Gamma^f$ . Given the outward normal velocity  $v_\Gamma^f$  on  $\Gamma$ , let  $\widehat{\mathcal{SH}} v_\Gamma^f$  be the  $(a^f, b^f)$ -discrete harmonic extension given by the solution of (21) with the boundary condition  $\widehat{\mathcal{SH}} v_\Gamma^f \cdot \boldsymbol{\tau} = 0$ . For both  $\mathcal{SH}$  and  $\widehat{\mathcal{SH}}$  are imposed essential boundary condition  $v_\Gamma^f$  for the normal component on  $\Gamma$ . The difference between them is in how the boundary condition is imposed for the tangential component on  $\Gamma$ . For the  $\mathcal{SH}$  is imposed homogeneous natural boundary condition, while for  $\widehat{\mathcal{SH}}$  is imposed homogeneous essential boundary condition. Both extensions,  $\mathcal{SH}_{\alpha^f}$  and  $\widehat{\mathcal{SH}}$ , satisfy the zero discrete divergence and boundary conditions in (21). Using this fact and the minimization property of the*

<p>1. Initialize</p> $x^{(0)} = 0 \quad (\text{No coarse problem})$ $\lambda^{(0)} = c$ <p>2. Iterate <math>k = 1, 2, \dots</math> until convergence</p> <p style="padding-left: 20px;">Precondition: <math>y^{(k-1)} = (NP)^\dagger (SP)(NP^T)^\dagger d^{(k-1)}</math></p> $\beta^k = \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle y^{(k-2)}, d^{(k-1)} \rangle \quad [\beta^{(1)} = 0]$ $r^{(k)} = y^{(k-1)} + \beta^{(k)} r^{(k)} \quad [r^{(1)} = y^{(0)}]$ $\alpha^{(k)} = \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle d^{(k)}, F r^{(k)} \rangle$ $x^{(k)} = x^{(k-1)} + \alpha^{(k)} r^{(k)}$ $d^{(k)} = d^{(k-1)} - \alpha^{(k)} F r^{(k)}$
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Figure 2: Implementation of the preconditioned conjugate gradient algorithm for the system (26) involving the FETI preconditioner (33).

$(a_{\alpha^f}^f, b^f)$ -discrete harmonic extension  $\mathcal{SH}_{\alpha^f}$  and the  $(a^f, b^f)$ -discrete harmonic extension  $\widehat{\mathcal{SH}}$  we get

$$\begin{aligned}
 a^f(\mathcal{SH}v_\Gamma^f, \mathcal{SH}v_\Gamma^f) &= a_0^f(\mathcal{SH}v_\Gamma^f, \mathcal{SH}v_\Gamma^f) \quad (\text{by definition}) \\
 &\leq a_0^f(\mathcal{SH}_{\alpha^f}v_\Gamma^f, \mathcal{SH}_{\alpha^f}v_\Gamma^f) \quad (\text{by the minimization property of } \mathcal{SH}) \\
 &\leq a_{\alpha^f}^f(\mathcal{SH}_{\alpha^f}v_\Gamma^f, \mathcal{SH}_{\alpha^f}v_\Gamma^f) \quad (\alpha^f > 0) \\
 &\leq a_{\alpha^f}^f(\widehat{\mathcal{SH}}v_\Gamma^f, \widehat{\mathcal{SH}}v_\Gamma^f) \quad (\text{by the minimization property of } \mathcal{SH}_{\alpha^f}) \\
 &= a_0^f(\widehat{\mathcal{SH}}_0v_\Gamma^f, \widehat{\mathcal{SH}}_0v_\Gamma^f) \quad (\text{because } \widehat{\mathcal{SH}}u^f \cdot \tau^f = 0 \text{ on } \Gamma) \\
 &\asymp \nu \|v_\Gamma^f\|_{H_{00}^{1/2}(\Gamma)}^2 \\
 &\asymp a^f(\mathcal{SH}_0v_\Gamma^f, \mathcal{SH}v_\Gamma^f).
 \end{aligned}$$

The last two equivalences follow from properties of the  $(a^f, b)$ -discrete harmonic extensions  $\mathcal{SH}$  and  $\widehat{\mathcal{SH}}$  (which coincides with the discrete Stokes harmonic extension); see [26, 38]. The two equivalences appearing above are independent of the permeability, fluid viscosity and mesh sizes. Then, the energy of the  $(a_{\alpha^f}^f, b)$ -discrete harmonic extensions is equivalent to the energy of the  $(a^f, b)$ -discrete harmonic extension, i.e., the discrete Stokes harmonic extension. This equivalence guarantees the extensions of Theorems 2, 4 and 7 to the case  $\alpha^f > 0$ .

## 8 Numerical Results

In this section we present numerical tests in order to verify the estimates in Theorems 2, 4 and 7. We consider  $\Omega^f = (1, 2) \times (0, 1)$  and  $\Omega^p = (0, 1) \times (0, 1)$ . See [11] and [25] for examples of exact solutions and compatible divergence and boundary data. Note that the reduced systems (15) and (26) involve only degrees of freedom on the interface  $\Gamma$ . In our test problems we compute the

eigenvalues of the preconditioned operators.

To solve both reduced systems (15) and (26) we can use the PCG algorithms described in Figures 1 and 2. Recall that the original system (11) is a “three times” saddle point problem. Note that since the finite element basis of  $M_{h^f}^f \times M_{h^p}^p$  and  $\Lambda^{h^p}$  have no zero mean, the finite element matrix in (13) has the kernel composed by constant pressures in  $\Omega = \text{int}(\bar{\Omega}^f \cup \bar{\Omega}^p)$  and constant Lagrange multipliers on  $\Gamma$ . The corresponding system is solved up to a constant pressure and a constant Lagrange multiplier. These constants can be recovered when imposing the zero average pressure constraint; see [25].

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-0}$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1, 1.0189(3)	1, 1.0198(3)	1, 1.0194(3)	1, 1.0193(3)	1, 1.0193(3)
$2^{-1} * 2^{-1}$	1, 1.0209(3)	1, 1.0200(3)	1, 1.0197(3)	1, 1.0196(3)	1, 1.0196(3)
$2^{-1} * 2^{-2}$	1, 1.0217(3)	1, 1.0205(3)	1, 1.0202(3)	1, 1.0201(3)	1, 1.0201(3)
$2^{-1} * 2^{-3}$	1, 1.0220(3)	1, 1.0208(3)	1, 1.0204(3)	1, 1.0203(3)	1, 1.0203(3)
$2^{-1} * 2^{-4}$	1, 1.0221(3)	1, 1.0209(3)	1, 1.0205(3)	1, 1.0204(3)	1, 1.0204(3)

Table 1: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here  $\kappa = 1$  and  $\alpha^f = 0$ .

### 8.1 BDD preconditioner

In the case of the BDD preconditioner (29) for (15), we solve a coarse problem before reducing the system to ensure balanced velocities at the beginning of the CG iterations.

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1, 21.0147(3)	1, 20.6035(3)	1, 20.3686(3)	1, 20.2893(3)
$2^{-1} * 2^{-1}$	1, 21.3303(6)	1, 20.8549(7)	1, 20.6550(7)	1, 20.5836(7)
$2^{-1} * 2^{-2}$	1, 22.0017(6)	1, 21.3392(9)	1, 21.1424(10)	1, 21.0735(10)
$2^{-1} * 2^{-3}$	1, 22.2367(6)	1, 21.6045(10)	1, 21.3626(9)	1, 21.2955(10)
$2^{-1} * 2^{-4}$	1, 22.3479(6)	1, 21.7006(10)	1, 21.4666(11)	1, 21.3929(9)

Table 2: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here  $\kappa = 10^{-3}$  and  $\alpha^f = 0$ .

We consider  $\alpha^f = 0$  and  $\nu = 1$ , and different values of  $h^f$  and  $h^p$  with non-matching grids across the interface  $\Gamma$ ; see Table 1 for the results when  $\kappa = 1$ , Table 2 for  $\kappa = 10^{-3}$  and Table 3 for the case  $\kappa = 10^{-5}$ . These three tables reveal growth of order  $O(1 + \frac{1}{\kappa})$  in  $\kappa$  and hence, verify the sharpness of the estimate in Theorem 2.

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1, 1977.08(3)	1, 1945.05(3)	1, 1932.10(3)	1, 1928.32(3)
$2^{-1} * 2^{-1}$	1, 1997.27(6)	1, 1972.77(7)	1, 1961.34(7)	1, 1957.88(7)
$2^{-1} * 2^{-2}$	1, 2053.57(6)	1, 2021.03(13)	1, 2010.27(17)	1, 2006.90(17)
$2^{-1} * 2^{-3}$	1, 2079.68(6)	1, 2044.05(13)	1, 2032.42(21)	1, 2029.13(31)
$2^{-1} * 2^{-4}$	1, 2090.10(6)	1, 2054.33(13)	1, 2042.26(22)	1, 2038.90(28)

Table 3: Minimum and maximum eigenvalues for the BDD preconditioned operator. Here  $\kappa = 10^{-5}$  and  $\alpha^f = 0$ .

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.0000, 1.0208(3)	1.0000, 1.0194(3)	1.0000, 1.0193(3)	1.0000, 1.0193(3)
$2^{-1} * 2^{-1}$	1.0017, 1.0200(3)	1.0000, 1.0197(3)	1.0000, 1.0196(3)	1.0000, 1.0196(3)
$2^{-1} * 2^{-2}$	1.0026, 1.0205(3)	1.0004, 1.0202(3)	1.0000, 1.0200(3)	1.0000, 1.0201(3)
$2^{-1} * 2^{-3}$	1.0027, 1.0208(3)	1.0007, 1.0204(3)	1.0001, 1.0203(3)	1.0000, 1.0203(3)
$2^{-1} * 2^{-4}$	1.0028, 1.0209(2)	1.0007, 1.0205(3)	1.0002, 1.0204(3)	1.0000, 1.0204(3)
$2^{-1} * 2^{-5}$	1.0028, 1.0209(2)	1.0007, 1.0206(3)	1.0002, 1.0205(3)	1.0000, 1.0204(3)

Table 4: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here  $\kappa = 1$  and  $\alpha^f = 0$ .

### 8.2 FETI preconditioner

In the case of the FETI preconditioner (33), we solve the reduced system (26) up to a constant Lagrange multiplier and a constant pressure. These constants are recovered after enforcing zero mean pressure on  $\Omega = \text{int}(\bar{\Omega}^f \cup \bar{\Omega}^p)$ ; see [25]. We recall that the FETI method can be viewed as the dual preconditioner counterpart of the BDD preconditioner. We repeat the same experiments mentioned above for the latter preconditioner.

We consider  $\alpha^f = 0$ ,  $\nu = 1$  and different values of  $h^f$  and  $h^p$  with nonmatching grids across the interface  $\Gamma$ ; see Table 4 for the results when  $\kappa = 1$ , Table 5 for  $\kappa = 10^{-3}$  and Table 6 for the case  $\kappa = 10^{-5}$ . Note that in Tables 4, 5 and 6 the minimum eigenvalues are strictly greater than one when  $h^f \leq 2h^p$ , and the value of the minimum eigenvalues seem to stabilize very quick for smaller  $h^f$  with fixed  $h^p$ . This confirms the extension of Theorem 7 for the case where  $h^f \leq 2h^p$ ; see Remark 8. In Table 7 we present the numerical results where one

$h_f \downarrow h_p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.000, 20.7608(3)	1.000, 20.4405(3)	1.000, 20.3110(3)	1.000, 20.2732(3)
$2^{-1} * 2^{-1}$	2.707, 20.9627(5)	1.000, 20.7177(7)	1.000, 20.6034(7)	1.000, 20.5688(7)
$2^{-1} * 2^{-2}$	3.634, 21.5257(5)	1.425, 21.2003(10)	1.000, 21.0927(12)	1.000, 21.0590(12)
$2^{-1} * 2^{-3}$	3.714, 21.7868(5)	1.651, 21.4305(9)	1.106, 21.3142(11)	1.000, 21.2813(12)
$2^{-1} * 2^{-4}$	3.760, 21.891(5)	1.663, 21.5333(9)	1.162, 21.4126(11)	1.026, 21.3790(12)
$2^{-1} * 2^{-5}$	3.771, 21.937(5)	1.673, 21.5768(9)	1.164, 21.4561(11)	1.040, 21.4220(12)

Table 5: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here  $\kappa = 10^{-3}$  and  $\alpha^f = 0$ .

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 1945.05(3)	1.00, 1932.10(3)	1.00, 1928.32(3)
$2^{-1} * 2^{-1}$	1.00, 1972.77(7)	1.00, 1961.34(7)	1.00, 1957.88(7)
$2^{-1} * 2^{-2}$	43.45, 2021.03(11)	1.00, 2010.27(17)	1.00, 2006.90(17)
$2^{-1} * 2^{-3}$	66.10, 2044.05(11)	11.58, 2032.42(20)	1.00, 2029.13(37)
$2^{-1} * 2^{-4}$	67.29, 2054.33(10)	17.20, 2042.26(19)	3.64, 2038.90(35)
$2^{-1} * 2^{-5}$	68.32, 2058.68(10)	17.42, 2046.61(10)	5.04, 2043.20(36)

Table 6: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here  $\kappa = 10^{-5}$  and  $\alpha^f = 0$ .

of the meshes on the interface is a refinement of the other side triangulation on the interface. We observe a behavior similar to the behavior of Table 6 with a bigger value for the minimum eigenvalue when  $h_f \leq h_p$ . This verifies the estimates of Theorem 7. This shows that the FETI preconditioner is scalable for the parameters faced in practice, i.e., the fluid side mesh finer than the porous side mesh and a small permeability  $\kappa$ . We conclude that the numerical experiments concerning the FETI preconditioner reveal the sharpness of the results obtained in Theorems 4 and 7 and Remark 8.

$h^f \downarrow h^p \rightarrow$	$2^{-1} * 2^{-1}$	$2^{-1} * 2^{-2}$	$2^{-1} * 2^{-3}$	$2^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 2002.47	1.00, 1961.35	1.00, 1937.86	1.00, 1929.93
$2^{-1} * 2^{-1}$	690.43, 2034.03	1.00, 1986.49	1.00, 1966.50	1.00, 1959.36
$2^{-1} * 2^{-2}$	627.36, 2101.17	176.56, 2034.92	1.00, 2015.24	1.00, 2008.35
$2^{-1} * 2^{-3}$	639.68, 2124.67	151.62, 2061.45	44.91, 2037.26	1.00, 2030.55
$2^{-1} * 2^{-4}$	642.44, 2135.79	154.45, 2071.06	38.04, 2047.66	11.98, 2040.29
$2^{-1} * 2^{-5}$	643.47, 2140.73	154.86, 2075.43	38.73, 2051.91	10.20, 2044.66

Table 7: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here  $\kappa = 10^{-5}$  and  $\alpha^f = 0$ . The refinement condition of Theorem 7 is satisfied under the diagonal.

Recall that we have assumed  $\alpha^f = 0$ . Now consider  $\alpha^f > 0$ . Numerical experiment were performed with  $\alpha^f > 0$  revealing results similar to the ones presented above for the case  $\alpha^f = 0$ . We only include Table 8 which shows the extreme eigenvalues of the FETI preconditioned operator for the case  $\alpha^f = 1$ ,  $\nu = 1$  and  $\kappa = 10^{-5}$ . This table presents a similar behavior to the one with  $\alpha^f = 0$  in Table 6 and hence, confirms Remark 10 which says that the parameter  $\alpha^f$  does not play much role for preconditioning.

## 9 On the multidomain case

The methods introduced in the previous sections considered only the two-subdomain case where discrete Stokes and Darcy indefinite subproblems are solved exactly in each subdomain and in each CG iteration. This might be very

$h^f \downarrow h^p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 1705.47	1.00, 1678.07	1.00, 1666.84	1.00, 1663.55
$2^{-1} * 2^{-1}$	162.74, 1814.26	1.00, 1787.53	1.00, 1776.50	1.00, 1773.22
$2^{-1} * 2^{-2}$	251.56, 1843.50	41.65, 1812.69	1.00, 1801.61	1.00, 1798.29
$2^{-1} * 2^{-3}$	267.47, 1849.46	63.63, 1816.43	11.24, 1804.66	1.00, 1801.34
$2^{-1} * 2^{-4}$	272.29, 1850.65	66.82, 1817.38	16.75, 1805.30	3.58, 1801.91
$2^{-1} * 2^{-5}$	273.34, 1851.08	67.99, 1817.68	17.37, 1805.57	4.97, 1802.14

Table 8: Minimum and Maximum eigenvalues of the FETI preconditioned operator. Here  $\kappa = 10^{-5}$  and  $\alpha^f = 1$ .

costly for large large subproblems since direct solvers or an accurate iterative solvers for the indefinite have to be used. In this Section we show that the methodology developed for the two-subdomain case can be developed also for the multidomain case. A complete analysis (using tools developed in Section 7) and extensive numerical experiments for the multidomain case will be object of future research and presented elsewhere.

Figure 3: Global interface  $\tilde{\Gamma}$  that includes all local interfaces and the Stokes/Darcy interface  $\Gamma$ .

We consider only the FETI method of Section 7 and the matching grids case. Assume that  $\mathcal{T}_{h^f}^f$  and  $\mathcal{T}_{h^p}^p$  coincide on the interface  $\Gamma$ . Let  $\{\Omega_j^i\}_{j=1}^{n^i}$  be a geometrically conforming substructures of  $\Omega^i$ ,  $i = f, p$ . We also assume that  $\{\Omega_j^f\}_{j=1}^{n^f} \cup \{\Omega_j^p\}_{j=1}^{n^p}$  form a geometrically conforming decomposition of  $\Omega$ , hence, the two decomposition are aligned on the interface  $\Gamma$ . We define the local inner interfaces as  $\Gamma_j^i = \partial\Omega_j^i \setminus \partial\Omega^i$ ,  $j = 1, \dots, n^i$ ,  $i = f, p$ . We also define

$$\tilde{\Gamma} = \bigcup_{j=1}^{n^f} \Gamma_j^f \cup \bigcup_{j=1}^{n^p} \Gamma_j^p \cup \Gamma.$$

See Figure 3. In order to simplify the presentation we assume for the fluid region, the spaces  $\mathbf{X}_{h^f}^f$  and  $M_{h^f}^f$  are the  $P2/P0$  triangular finite elements while for the porous region, the spaces  $\mathbf{X}_{h^p}^p \subset \mathbf{X}^p$  and  $M_{h^p}^p \subset L^2(\Omega^p)$  are the lowest order Raviart-Thomas finite elements based on triangles. Similar as in the previous sections, we decompose the velocity and pressure spaces as follows:

- $\mathbf{X}_I^f$ : interior velocities in the subdomains  $\{\Omega_j^f\}_{j=1}^{n^f}$ .
- $\mathbf{X}_{\tilde{\Gamma}}^f$ : interface velocities on  $\tilde{\Gamma} \cap \bar{\Omega}^f$ .
- $\mathbf{X}_I^p$ : interior velocities in the subdomains  $\{\Omega_j^p\}_{j=1}^{n^p}$ .
- $\mathbf{X}_{\tilde{\Gamma}}^p$ : interface velocities on  $\tilde{\Gamma} \cap \bar{\Omega}^p$ .

- $M_I^i$ , ( $i = p, f$ ): interior zero mean pressure in each subdomain  $\{\Omega_j^i\}_{j=1}^{n^i}$ ,  $i = f, p$ .
- $M_0^i$ , ( $i = p, f$ ): constant pressure in each subdomain  $\{\Omega_j^i\}_{j=1}^{n^i}$ ,  $i = f, p$ .
- $M_I = M_I^f \times M_I^p$
- $\mathbf{X}_I = \mathbf{X}_I^f \times \mathbf{X}_I^p$ ,  $\mathbf{X}_{\tilde{\Gamma}} = \mathbf{X}_{\tilde{\Gamma}}^f \times \mathbf{X}_{\tilde{\Gamma}}^p$ ,  $M_I = M_I^f \times M_I^p$  and  $M_0 = M_0^f \times M_0^p$ .

After imposing the mortar condition as in Section 4 we can reduce (11) to a Schur complement system on the interface  $\tilde{\Gamma}$ ,

$$\tilde{S}_{\tilde{\Gamma}} \mathbf{u}_{\tilde{\Gamma}} = \tilde{b}_{\tilde{\Gamma}} \quad (39)$$

which is the multidomain generalization of the reduced system (15).

The  $\tilde{\Gamma}$ -interface velocity space  $\mathbf{X}_{\tilde{\Gamma}}$  can be decompose in primal and dual degrees of freedom, i.e.,  $\mathbf{X}_{\tilde{\Gamma}} = \mathbf{X}_C \oplus \mathbf{X}_{\Delta}$  where  $\mathbf{X}_C$  includes the function which are continuous with respect to the primal degrees of freedom. The primal variables for the fluid velocity field are the continuity of both components of the fluid velocities at the corners and the continuity of the mean normal flux on each face of the subdomains  $\{\Omega_j^f\}_{j=1}^{n^f}$ . For the the porous side, the primal variables are the continuity of the mean normal flux on the each face of the subdomains  $\{\Omega_j^p\}_{j=1}^{n^p}$ ; see [30, 31, 32, 41]. For faces of the subdomains on  $\Gamma$ , only the continuity of the mean fluxes are imposed. . The space  $\mathbf{X}_{\Delta}$  includes the remaining fluid side velocity degrees of freedom and the remaining porous media velocity degrees of freedom.

Functions in  $\mathbf{X}_{\Delta}$  do not satisfy the continuity requirements in each region and across interface  $\tilde{\Gamma} \cap \Omega^p$  on the the Darcy side (the slave side). These continuity requirements can be enforced using a Lagrange multiplier  $\tilde{\lambda}$  define on  $\tilde{\Gamma}$  by the equation

$$B_{\Delta} \mathbf{v}_{\Delta} = 0.$$

We ensure that this condition coincides with the last equation of (13) that corresponds to the flux continuity across the Stokes/Darcy interface  $\Gamma$ . On the Stokes/Darcy interface  $\Gamma$  we use the same Lagrange multiplier of the dual formulation (26). Proceeding as in [30] we can obtain a reduced system of the form

$$\tilde{F} \tilde{\lambda} = \tilde{b}$$

which corresponds to the multidomain version of (26). The preconditioner operator is of the form

$$B_{\Delta} \tilde{S}_{\tilde{\Gamma}} B_{\Delta}^T,$$

where  $\tilde{S}_{\tilde{\Gamma}}$  was introduced in (39).

## 10 Conclusions and Final Comments

We consider the problem of coupling fluid flows with porous media flows with Beavers-Joseph-Saffman condition on the interface. We choose a discretization consisting of Taylor-Hood finite elements of order two on the free fluid side and the lowest order Raviart-Thomas finite element on the porous fluid side. The meshes are allowed to be nonmatching across the interface.

We design and analyze two preconditioners for the resulting symmetric linear system. We note that the original linear system is symmetric indefinite and involves three Lagrange multipliers: one for each subdomain pressure and a third one to impose the weak conservation of mass across the interface  $\Gamma$ ; see Section 1.

One preconditioner is based on BDD methods and the other one is based on FETI methods. In the case of the BDD preconditioner, the energy is controlled by the Stokes side, while in the FETI preconditioner, the energy is controlled by the Darcy system; see Theorems 2 and 4. In both cases a bound  $C_1(\frac{\kappa+1}{\kappa})$  is derived. Furthermore, under the assumption that the fluid side mesh on the interface is finer than the corresponding porous side mesh, we derive the better bound  $C_2(\frac{\kappa+1}{\kappa+(h^p)^{\frac{1}{2}}})$  for the FETI preconditioner; see Theorem 7 and Remark 8. This better bound also shows that the FETI preconditioner is more scalable for parameters faced in practice, e.g., problems with small permeability  $\kappa$  and where the fluid side mesh is finer than the porous side mesh. The constants  $C_1$  and  $C_2$  above are independent of the fluid viscosity  $\nu$ , the mesh ratio across the interface, and the permeability  $\kappa$ .

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