Shifted Laplacian RAS Solvers for the Helmholtz Equation

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1 Introduction

We consider the Helmholtz equation:

$$\begin{gathered} -\Delta u^* - k^2 u^* = f \quad \text{in} \quad \Omega \\ u^* = g_D \text{ on } \partial\Omega_D,\ \frac{\partial u^*}{\partial n} = g_N \text{ on } \partial\Omega_N,\ \frac{\partial u^*}{\partial n} + iku^* = g_S \text{ on } \partial\Omega_S \end{gathered} \tag{1}$$

where Ω is a bounded polygonal region in $\Re^2$, and the $\partial\Omega_D$, $\partial\Omega_N$ and $\partial\Omega_S$ correspond to subsets of $\partial\Omega$ where the Dirichlet, Neumann and Sommerfeld boundary conditions are imposed.

The main purpose of this paper is to introduce novel two-level overlapping Schwarz methods for solving the Helmholtz equation. Among the most effective parallel two-level domain decomposition solvers for the Helmholtz equation on general unstructured meshes, we mention the FETI-H method introduced by Farhat et al. [2000], and the WRAS-H-RC method introduced by Kimn and Sarkis [2007]. FETI-H type preconditioners belong to the class of nonoverlapping domain decomposition methods. FETI-H methods can be viewed as a modification of the original FETI method introduced by Farhat et al. [1994]. The local solvers in FETI-H are based on Sommerfeld boundary conditions, see Deprés [1991], while the coarse problem is based on plane waves. WRAS-H-RC type preconditioners belong to the class of overlapping Schwarz methods. They can be viewed as a miscellaneous of several methods to enhance the effectiveness of the solver for Helmholtz problems. The first ingredient of WRAS-H-RC preconditioners is the use of Sommerfeld boundary conditions for the local solvers on overlapping subdomains. This idea is

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similar to what was done in FETI-H, however, now for the overlapping case. This idea can be found for instance in the work of Cai et al. [1998] and Kimn [2005]. The second ingredient is the use of the Weighted Restricted Additive Schwarz (WRAS) method introduced by Cai and Sarkis [1999] in order to average the local overlapping solutions. The third ingredient is the use of partition of unity coarse spaces, see Sarkis [2002]. Here we consider the multiplication of a partition of unity times plane waves; see Melenk [1995]. The fourth ingredient is how to define the coarse problem. It was discovered in Kimn and Sarkis [2007] that a dramatic gain in performance can be obtained if WRAS techniques are applied to the fine-to-coarse restriction operator and the coarse-to-fine prolongation operator. The idea is to force the to act more locally on the fine-to-coarse transference of information and globally on the coarse-to-fine phase. The last ingredient is to put all these pieces together. The idea is to extend the Balancing Domain Decomposition (BDD) methods of Mandel [1993], which were originally developed for the nonoverlapping case, to the overlapping case. This extension was introduced in Kimn and Sarkis [2006] and the methods there were denoted by Overlapping Balancing Domain Decomposition (OBDD) methods. The WRAS-H-RC methods in Kimn and Sarkis [2007] stand for "WRAS" for the local solvers, "H" for the FETI-H ingredients included in the methods, and "RC" for the restricted flavor of coarse problem.

Here in this paper we investigate numerically new techniques to improve further the performance of the WRAS-H-RC. More precisely, the shifted Laplacian techniques introduced in Gijzen et al. [2007] and Erlangga et al. [2004], are used to construct novel local solvers. We investigate how the various kinds of shifts affect the performance of the algorithms. As a result, we discover novel preconditioners that are more effective than the existing ones.

2 Discrete Formulation of the Problem

From a Green's formula, (1) can be reduced to: Find $u^* - u_D^* \in H_D^1(\Omega)$ such that,

$$\begin{aligned} a(u^*, v) &= \int_\Omega (\nabla u^* \cdot \nabla \bar{v} - k^2 u^* \bar{v})\, dx + ik \int_{\partial\Omega_S} u^* \bar{v}\, ds \\ &= \int_\Omega f \bar{v}\, dx + \int_{\partial\Omega_N} g_N \bar{v}\, ds + \int_{\partial\Omega_S} g_S \bar{v} = F(v),\ \forall v \in H_D^1(\Omega), \end{aligned} \tag{2}$$

where u_D^* is an extension of g_D to $H^1(\Omega)$, and $H_D^1(\Omega)$ is the space of $H^1(\Omega)$ functions vanishing on $\partial\Omega_D$.

Let $\mathcal{T}_h(\Omega)$ be a quasi-uniform triangulation of Ω and let $V \subset H_D^1(\Omega)$ be the finite element space of continuous piecewise linear functions vanishing on

$\partial\Omega_D$. We assume that g_D on $\partial\Omega_D$ is a piecewise linear continuous function on $\mathcal{T}^h(\partial\Omega_D)$ and we have eliminated g_D by a discrete trivial zero extension inside Ω. We then obtain a discrete problem of the following form: Find $u \in V$ such that

$$a(u,v) = f(v), \quad \forall\ v \in V. \tag{3}$$

Using the standard hat basis functions, (3) can be rewritten as a linear system of equations of the form

$$Au = f. \tag{4}$$

3 Description of the WRAS-H-RC Methods

3.1 Partitioning and Subdomains

Given the triangulation $\mathcal{T}^h(\Omega)$, we assume that a domain partition by elements has been applied and resulted in N nonoverlapping subdomains $\Omega_i, i = 1, \ldots N$, such that

$$\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i \ \text{ and } \ \Omega_i \cap \Omega_j = \emptyset, \ \text{ for } \ j \neq i.$$

Let δ be a nonnegative integer. Define $\Omega_i^0 = \Omega_i$. For $\delta \geq 1$, define the overlapping subdomains Ω_i^δ as follows: let Ω_i^1 be the one-overlap element extension of Ω_i^0 by including all the immediate neighboring elements $\tau_h \in \mathcal{T}^h(\Omega)$ such that $\overline{\tau}_h \cap \overline{\Omega}_i^0 \neq \emptyset$. Using this idea recursively, we can define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i = \Omega_i^0 \subset \Omega_i^1 \subset \cdots \subset \Omega_i^\delta \cdots$$

3.2 Partition of the Unity

Let w be a nonnegative integer. For nodes x on $\partial\Omega_i^0$ define $\hat{\vartheta}_i^w(x) = 1$, for nodes x on $\partial\Omega_i^1 \backslash \overline{\Omega}_i^0$ define $\hat{\vartheta}_i^w(x) = 1 - 1/(w+1)$, for nodes x on $\partial\Omega_i^2 \backslash \overline{\Omega}_i^1$ define $\hat{\vartheta}_i^w(x) = 1 - 2/(w+1)$, and recursively until $\hat{\vartheta}_i^w(x) = 0$. For nodes x in $\overline{\Omega} \backslash \overline{\Omega}_i^w$ define $\hat{\vartheta}_i^w(x) = 0$. The partition of unity ϑ_i^w is defined as

$$\vartheta_i^w = I_h\left(\frac{\hat{\vartheta}_i^w}{\sum_{j=1}^{N} \hat{\vartheta}_j^w}\right) \quad i = 1, \cdots, N,$$

where I_h is the nodal piecewise linear interpolant on $\mathcal{T}^h(\overline{\Omega})$. Note that the support of ϑ_i^w is Ω_i^{w+1} and $|\nabla \vartheta_i^w| \leq O((w+1)/h)$. We define the weighting diagonal matrix D_i^w as equal to $\vartheta_i^w(x)$ at the nodes x of $\overline{\Omega}$.

3.3 Local Problems

Let us denote by V_i^δ, $i = 1, \cdots, N$, the local space of functions in $H^1(\Omega_i^\delta)$ which are continuous piecewise linear and vanishes only on $\partial\Omega_i^\delta \cap \partial\Omega_D$. For each subdomain Ω_i^δ, let $R_i^\delta : V \to V_i^\delta$ be the regular restriction operator on V_i^δ, that is, $v_i(x) = v(x)$ for nodes $x \in \overline{\Omega}_i^\delta$.

For the local solvers, we respect the original boundary condition and impose Sommerfeld boundary condition on the interior boundaries $\partial\Omega_i^\delta \backslash \partial\Omega$. The associated local projections in matrix form are defined by

$$T_{i,WRAS-H}^\delta = (R_i^\delta D_i^\delta)^T (\tilde{A}_i^\delta)^{-1} R_i^\delta A \qquad i = 1, \cdots, N \tag{5}$$

where $\tilde{A}_i^\delta$ are the matrix form of

$$\tilde{a}_i^\delta(u_i, v_i) = \int_{\Omega_i^\delta} (\nabla u_i \cdot \nabla \overline{v}_i - k^2 u_i \overline{v}_i)\, dx + ik \int_{\partial\Omega_i^\delta \backslash (\partial\Omega_D \cup \partial\Omega_N)} u_i \overline{v}_i \, ds. \tag{6}$$

3.4 Coarse Problem

Let c be a nonnegative integer. The coarse space $V_0^{c,p} \in V$ is defined as the space spanned by $D_i^c Q_j^D$ for $i = 1, \ldots, N$ and $j = 1, \cdots, p$. Here, $Q_j := e^{ik\eta_j^T x}$, where $\eta_j = (\cos(\theta_j), \sin(\theta_j))$, with $\theta_j = (j-1) \times \frac{\pi}{p}, j = 1, \cdots, p$, while $Q_j^D(x) := Q_j(x)$ for nodes $x \in \overline{\Omega} \backslash \partial\Omega_D$ and $Q_j^D(x) := 0$ for nodes x on $\partial\Omega_D$. The coarse-to-fine prolongation matrix $(E_0^{c,p})$ consists of columns $D_i^\delta Q_j^D$, while the fine-to-coarse restriction matrix $R_0^{\delta,p}$ consists of rows $(R_i^\delta)^T R_i^\delta Q_j^D$. The first coarse problem we consider in this paper is given by

$$P_{0,RC}^{\delta,c,p} = E_0^{c,p} [R_0^{\delta,p} A E_0^{c,p}]^{-1} R_0^{\delta,p}. \tag{7}$$

3.5 Hybrid Preconditioners

The first preconditioner we consider is given by

$$T_{WRAS-H-RC}^{\delta,c,p} := P_{0,RC}^{\delta,c,p} + (I - P_{0,RC}^{\delta,c,p}) (\sum_{i=1}^{N} T_{i,WRAS-H}^\delta) (I - P_{0,RC}^{\delta,c,p}). \tag{8}$$

Because $P_{0,RC}^{\delta,c,p}$ is a projection, only one coarse problem solver is necessary per iteration of the iterative method.

Other hybrid preconditioners can also be designed. For instance, we can replace the local problem $T^{\delta}_{i,WRAS}$ by

$$P^{\delta}_{i,OBDD-H} := (R^{\delta}_i D^{\delta}_i)^T (\tilde{A}^{\delta}_i)^{-1} R^{\delta}_i D^{\delta}_i A$$

or/and replace the coarse problem $P^{\delta,c,p}_{0,RC}$ by something more classical such as

$$P^{c,p}_0 = E^{c,p}_0 [(E^{c,p}_0)^T A E^{c,p}_0]^{-1} (E^{c,p}_0)^T .$$

Inserting these operators properly into (7) we obtain preconditioners which we denote by $T^{\delta,c,p}_{WRAS-H}$, $T^{\delta,c,p}_{OBDD-H}$ or $T^{\delta,c,p}_{OBDD-H-RC}$. An interesting structure that $T^{\delta,c,p}_{WRAS-H-RC}$ has, and the others do not, is that the same restriction operators R^{δ}_i are used to compute the right-hand side for both the local and coarse problems, therefore, computational efficiency can be explored.

4 Shifted Local Operators

The matrix $\tilde{A}^{\delta}_i$ obtained from the bilinear form (6) can be written as

$$\tilde{A}^{\delta}_i = A^{\delta}_i - k^2 M^{\delta}_i + ikB^{\delta}_i ,$$

where A^{δ}_i, M^{δ}_i, and B^{δ}_i are the corresponding matrices associated to

$$\int_{\Omega^{\delta}_i} \nabla u_i \cdot \nabla \overline{v}_i \, dx + ik \int_{\partial\Omega^{\delta}_i \cap \partial\Omega_S} u_i \overline{v}_i \, ds, \qquad \int_{\Omega^{\delta}_i} u_i \overline{v}_i \, dx \quad \text{and} \quad \int_{\partial\Omega^{\delta}_i \setminus \partial\Omega} u_i \overline{v}_i \, ds,$$

respectively. We note that the local matrix $A^{\delta}_i - k^2 M^{\delta}_i$ is singular if k^2 is a generalized eigenvalue of A^{δ}_i. Alternatively, if we enforce zero Dirichlet boundary condition on the interior boundaries $\partial\Omega_i \cap \Omega^{\delta}_i$, singularities also might occurs, specially when the subdomains are not small enough. The Sommerfeld term plays the rule of shifting the real spectrum of $A^{\delta}_i - k^2 M^{\delta}_i$ to the upper part of the complex plane, therefore, elliminating possible zero eigenvalues. More general shifts were introduced recently by Gijzen et al. [2007], Erlangga et al. [2004] to move the spectrum to a disk on the the first quadrant. Inspired by this work, we now consider shifts to define the local solvers as

$$\tilde{A}^{\delta}_i(\alpha_r, \alpha_i, \beta_r, \beta_i) = A^{\delta}_i + (\alpha_r + i\alpha_i) k^2 M^{\delta}_i + (\beta_r + i\beta_i) k B^{\delta}_i , \qquad (9)$$

that is, the local Laplacians A^{δ}_i are shifted by a complex combination of M^{δ}_i and B^{δ}_i. Note that $\tilde{A}^{\delta}_i(-1, 0, 0, 1)$ reduces to the original local solver (6), while $\tilde{A}^{\delta}_i(-1, 0, 0, 0)$ to $A^{\delta}_i - k^2 M^{\delta}_i$.

5 Numerical Results

As a numerical test, we consider a wave guided problem for solving the Helmholtz equation on the unit square. We consider homogeneous Neumann boundary condition on the horizontal sides, homogeneous Sommerfeld on the right vertical side, and a constant identical to one Dirichlet on the left vertical side. The stopping criteria for the PGMRES is to reduce the initial residual by a factor of 10^{-6}. In all tests the right preconditioner is applied.

The triangulation is composed of Courant elements of mesh size $h = 1/256$. The nonoverlapping subdomains Ω_i^0 are squares of size $1/M$, and the number of subdomains is denoted by $nsub = M \times M$. The pair (δ, c) refers to how many layers of elements are used to define the extension of the overlapping subdomains Ω_i^δ and the extension of the support of the coarse basis functions, respectively. The constant k refers to the wave number and p denotes the number of local plane waves used in the coarse space. Table 1 shows that the method $P_{WRAS-H-RC}$ is the most effective method among those introduced in Section 3.5. Table 2 shows that we should select the support for the coarse basis functions larger enough, larger than the size of the extended subdomains. Tables 1 and 2 show that the number of iterations decreases when we increase the size of the overlap.

We now test the effectiveness of $P_{WRAS-H-RC}$ for several combinations of local solvers $\tilde{A}_i^\delta(\alpha_r, \alpha_i, \beta_r, \beta_i)$. Table 3 shows results for $\delta = 2$ and Table 4 for $\delta = 0$. We can see from Tables 3 and 4 that the number of iterations using the original local problem are 13 and 34, respectively. It is very surprising and interesting to observe that the number of iterations are 9 and 18 for the combination $(0, 1, 1, 0)$, a respectable gain in efficiency. Tables 3 and 4 reveal that there exist more effective choices for local solvers rather than the common choice approach of adding a Sommerfeld term on the interior boundary of the subdomains. These preliminary results are very inspiring and encouraging for further numerical and theoretical investigations.

Table 1 The Guided Wave Problem, Sommerfeld boundary condition on interior subdomain boundaries, $n = 257$, $nsub = 64(8 \times 8)$, Tol=10^{-6}, $k = 20$

(δ, c, p)	(0,7,4)	(1,7,4)	(2,7,4)
$OBDD-H$	158	85	43
$WRAS-H$	150	74	36
$OBDD-H-RC$	40	23	16
$WRAS-H-RC$	34	19	13

Table 2 **WRAS-H-RC** The Guided Wave Problem, Sommerfeld boundary condition on interior subdomain boundaries, $n = 257$, $nsub = 64(8 \times 8)$, $p = 4$, Tol=10^{-6}, $k = 20$

	WRAS-H-RC							
$c=$	1	2	3	4	5	6	7	8
δ =0	78	67	54	46	40	37	34	32
δ =1	190	36	31	25	22	21	19	18
δ =2	181	181	19	18	16	14	13	12

Table 3 The Guided Wave Problem, **WRAS-H-RC** algorithm with Shifted Laplacian local problems, $n = 257$, $nsub = 64$, Tol=10^{-6}, $p = 4$, $k = 20$, $c = 7$, $\delta = 2$

	$\alpha_r =$	-1	-1	-1	0	0	0	1	1	1
	$\alpha_i =$	-1	0	1	-1	0	1	-1	0	1
$\beta_r = -1$	$\beta_i = -1$	37	53	116	22	28	210	17	22	48
$\beta_r = -1$	$\beta_i = 0$	236	123	199	154	275	139	105	300*	138
$\beta_r = -1$	$\beta_i = 1$	66	34	28	227	24	16	55	22	17
$\beta_r = 0$	$\beta_i = -1$	20	23	62	14	14	20	12	11	12
$\beta_r = 0$	$\beta_i = 0$	19	16	13	17	300*	12	14	13	10
$\beta_r = 0$	$\beta_i = 1$	55	**13**	13	23	13	11	15	12	11
$\beta_r = 1$	$\beta_i = -1$	15	12	12	13	10	10	12	10	9
$\beta_r = 1$	$\beta_i = 0$	13	17	11	12	10	9	12	10	**8**
$\beta_r = 1$	$\beta_i = 1$	17	10	11	12	10	9	11	10	9

Table 4 The Guided Wave Problem, **WRAS-H-RC** algorithm with Shifted Laplacian local problems, $n = 257$, $nsub = 64$, Tol=10^{-6}, $p = 4$, $k = 20$, $c = 7$, $\delta = 0$

	$\alpha_r =$	-1	-1	-1	0	0	0	1	1	1
	$\alpha_i =$	-1	0	1	-1	0	1	-1	0	1
$\beta_r = -1$	$\beta_i = -1$	168	213	300*	99	168	300*	69	106	300*
$\beta_r = -1$	$\beta_i = 0$	291	207	243	238	300*	209	221	300*	300*
$\beta_r = -1$	$\beta_i = 1$	300*	137	101	300*	130	63	300*	107	67
$\beta_r = 0$	$\beta_i = -1$	55	69	289	38	42	80	34	30	32
$\beta_r = 0$	$\beta_i = 0$	45	31	30	38	300*	27	34	24	24
$\beta_r = 0$	$\beta_i = 1$	279	**34**	33	94	39	30	40	35	31
$\beta_r = 1$	$\beta_i = -1$	34	31	39	29	25	22	27	24	21
$\beta_r = 1$	$\beta_i = 0$	27	22	21	**24**	20	**18**	24	21	20
$\beta_r = 1$	$\beta_i = 1$	51	23	21	25	21	20	23	21	21

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