A FETI-DP PRECONDITIONER FOR A COMPOSITE FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD

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Abstract. In this paper a Nitsche-type discretization based on discontinuous Galerkin (DG) method for an elliptic two-dimensional problem with discontinuous coefficients is considered. The problem is posed on a polygonal region Ω which is a union of N disjoint polygonal subdomains Ω̄i of diameter O(H̄i). The discontinuities of the coefficients, possibly very large, are assumed to occur only across the subdomain interfaces ∂Ω̄i. Inside each subdomain, a conforming finite element space on a quasiuniform triangulation with mesh size O(h̄i) is introduced. To handle the nonmatching meshes across the subdomain interfaces, a discontinuous Galerkin discretization is applied only on the interfaces. For solving the resulting discrete system, a FETI-DP type method is proposed and analyzed. It is established that the condition number of the preconditioned linear system is estimated by $C(1 + \max_i \log H̄_i/h̄_i)^2$ with a constant $C$ independent of $h̄_i$, $h̄_i/h̄_j$, $H̄_i$ and the jumps of coefficients. The method is well suited for parallel computations and it can be extended to three-dimensional problems. Numerical results are presented to validate the theory.

Key words. Interior penalty discretization, discontinuous Galerkin, elliptic problems with discontinuous coefficients, finite element method, FETI-DP algorithms, preconditioners

AMS subject classifications. 65F10, 65N20, 65N30

1. Introduction. In this paper we consider an elliptic problem with highly discontinuous coefficients. The problem is posed on a polygonal region Ω which is a union of disjoint 2-D polygonal subdomains Ω̄i of diameter O(H̄i) and we assume that this partition $\{\Omegā_i\}^N_{i=1}$ is geometrically conforming, i.e., ∀i ≠ j the intersection $\partial Ω̄_i \cap \partial Ω̄_j$ is empty or is a common corner or edge of Ω̄i and Ω̄j. The discontinuities of the coefficients are assumed to occur only across $\partial Ω̄_i$. The problem is approximated by a conforming finite element method (FEM) inside each Ω̄i, with $h̄_i$ as a mesh parameter, and nonmatching meshes are allowed to occur across the interfaces $\partial Ω̄_i$. In order to deal with the nonconformity of the FE spaces across $\partial Ω̄_i$, a discrete problem is formulated using the symmetric interior penalty DG method on the $\partial Ω̄_i$ only, see [33, 31]. This kind of composite discretization is motivated by the local regularity of the solution of the problem. This approach, which we denote by composite FE/DG method, falls in the class of Nitsche-type mortaring methods, see [4, 33, 20], where conforming FEMs can be explored inside the subdomains Ω̄i rather than using the full DG method everywhere, see [4, 11]. As a result, the composite FE/DG method approach will imply in essential computational savings both in terms of memory and CPU time. The main goal of this paper is to design and analyze a FETI-DP preconditioner for the resulting discrete system. To the best of our knowledge, the FETI-DP method has never been considered before in the literature for DG or composite FE/DG discretizations.

The first FETI-DP method for standard continuous finite element discretization

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was introduced in [15] and is a nonoverlapping domain decomposition method that enforces continuity of the solution at subdomain interfaces by Lagrange multipliers except at subdomain corners where the continuity is enforced directly by assigning a unique value for the functions at each corner. The first mathematical analysis of the method was provided in [30]. The method was further improved by enforcing the continuity directly on averages across the edges or faces on subdomain interfaces [16, 24, 34], resulting in better parallel scalability for three-dimensional problems. FETI-DP methods, and also methods of this class such as FETI, BDDC and BDD, have been tested very successfully and analyzed theoretically for a wide variety of problems and standard continuous discretizations, see [34] and references therein.

The main goal of this paper is to develop the FETI-DP methodology for a non-trivial problem using a composite FE/DG method. More specifically, we consider a scalar second order elliptic problem with discontinuous coefficients with nonmatching meshes across the subdomain interfaces. The discontinuities of the coefficients do not necessarily satisfy the quasi-monotonicity condition on the jumps of the coefficients, see [14]. In addition, the mesh size ratio \( h_i / h_j \) across the interfaces between two neighboring subdomains \( \Omega_i \) and \( \Omega_j \) are not necessary of \( O(1) \). In [11], two discretizations were introduced which are based on symmetric IPDG discretizations on the subdomain interfaces. These discretizations aim at solving problems in the presence of both nonmatching meshes and discontinuous coefficients across these interfaces. The first discretization is based on the fluxes that we get by integration by parts, while the second one the fluxes are modified using harmonic average of the coefficients. It turns out that the second discretization is easier for the preconditioning point of view since the nonsymmetric term of the bilinear form is smaller, hence it requires a smaller penalty term, and as a consequence, the latter term can be easily controled by the energy term of the bilinear form by using a proper Poincaré-Friedrich inequality. To the best of our knowledge, there is no literature on preconditioning for the first discretization, and this paper will fill this scientific gap. Furthermore, even for the second discretization, numerical results show, see [12, 13], that preconditioners based on Neumann-Neumann or BDDC with edge constraints require an interface mesh-coefficient condition to achieve robustness with respect to both jumps of coefficients and mesh sizes ratio across the interfaces. This condition is not needed for the preconditioner that we consider here in this paper. We expect that the methodology developed here can be extended to a large class of problems and for full DG discretizations, see [4, 21, 31]. In order to discuss why the development of FETI-DP preconditioners for composite FE/DG methods is more complicated compared to FETI-DP for conforming or mortar FE methods, we next give a brief introduction on FETI-DP for composite FE/DG method. A fully discussion and analysis of this method will be given in the remaining sections of this paper.

In this paper, the composite FE/DG discrete problem is reduced to the Schur complement problem with respect to unknowns on the interfaces of the subdomains \( \Omega_i \). For that, discrete harmonic functions defined in a special way, i.e., in the DG sense, are used. We note that there are unknowns on both sides of the interfaces \( \partial \Omega_i \), and on the contrary of the mortar methods, see [7], unknowns from one side of a interface cannot be eliminated via master-to-slave projections. This means that the unknowns on both sides of the interfaces should be kept as degrees of freedom of the linear system to be solved. Additionally, due to the DG penalty terms on the
interfaces $\partial \Omega_i$, these degrees of freedom are coupled across these interfaces. These issues characterize some of the main difficulties on designing and analyzing FETI-DP type methods for composite FE/DG discretizations. Distinctively from the classical conforming FE discretizations or from the mortar finite element discretizations, here a double layer of Lagrange multipliers are needed on interfaces rather than a single layer of Lagrange multipliers as normally is seen in FETI-DP for conforming or mortar FEMs. Despite the fact we follow the FETI-DP abstract approach from [34, 30], which was aimed to single layer of Lagrange multipliers, in this paper we successfully overcome this difficulty. The algorithm we develop in this paper is as follows. Let $\Gamma'_i$ be the union of all edges $\bar{F}_{ij}$ and $\bar{F}_{ji}$ which are common to $\Omega_i$ and $\Omega_j$, where $\bar{F}_{ij}$ and $\bar{F}_{ji}$ refer to the $\Omega_i$ and $\Omega_j$ sides, respectively, see Figure 3.1. We note that each $\Gamma'_i$ has unknowns (degrees of freedom) corresponding to nodal points on the closure $\partial \Omega_i \cup \partial \Omega$ and on the $\bar{F}_{ji} \subset \partial \Omega_j$. We now need to couple $\Gamma'_i$ with the other $\Gamma'_j$. We first impose continuity at the corners of $\Gamma'_i$ (which are corners of $\Omega_i$ and common endpoints of $\bar{F}_{ji}$) with the corners of the $\Gamma'_j$, see Figure 4.1. These unknowns are called primal. The remaining unknowns on $\Gamma'_i$ and $\Gamma'_j$ are called dual and have jumps, hence, Lagrange multipliers are introduced to eliminate these jumps, see Figure 3.2. For the dual system with Lagrange multipliers, an special block diagonal preconditioner is designed. It leads to independent local problems on $\Gamma'_i$ for $1 \leq i \leq N$. It is proved that the proposed method is almost optimal with a condition number estimate bounded by $C(1 + \max_i \log H_i/h_i)^2$, where $C$ does not depend on $h_i$, $h_j$, $h_i/h_j$, the number of subdomains $\Omega_i$ and the jumps in the coefficients. The method can be extended to composite FE/DG discretizations of three-dimensional problems, see Section 7. The FETI-DP method developed here complements the BDDC methodology for the composite FE/DG discretizations developed recently in [12]. We note that the discretization used there (based on the harmonic average of the coefficients) is not the same as the one we consider here, additionally, the constraints used there are based on edges while here are based on corners. We point out that the introduction of corner constraints eliminates the interface condition assumption required in [12]. We note that other types of preconditioners have been considered for solving DG discretizations. In connection with block diagonal or overlapping Schwarz methods see for example [17, 18, 27, 8, 1, 2, 10, 29, 5] while for multilevel preconditioners [19, 22, 28, 26, 25, 6]. We note that these latter preconditioners do not use discrete harmonic extensions and does not analyze the cases of nonmatching meshes across subdomains and discontinuous coefficients.

The paper is organized as follows. In Section 2 the differential problem and a composite FE/DG discretization are formulated. In Section 3, the Schur complement problem is derived using discrete harmonic functions defined in a special way (in the DG sense). In Section 4, the so-called FETI-DP method is introduced, i.e., the Schur complement problem is reformulated by imposing continuity for the primal variables and by using Lagrange multipliers at the dual variables, and an special block diagonal preconditioner is defined. The main results of the paper are Theorem 4.3 and Lemma 4.5. Section 5 is devoted to the implementation of the FETI-DP method while in Section 6 numerical tests are reported which confirm the theoretical results. In Section 7, conclusions and extensions of the proposed preconditioner are discussed.

2. Differential and discrete problems. In this section we discuss the continuous and discrete problems we take into consideration for preconditioning.
2.1. Differential problem. Consider the following problem: Find $u_{ex}^* \in H^1_0(\Omega)$ such that

\begin{equation}
(2.1) \quad a(u_{ex}^*, v) = f(v) \quad \forall v \in H^1_0(\Omega)
\end{equation}

where

$$a(u, v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_{\Omega} f v \, dx.$$ 

We assume that $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$ and the substructures $\Omega_i$ are disjoint shaped regular polygonal subregions of diameter $O(H_i)$. We assume the partition $\{\Omega_i\}_{i=1}^{N}$ is geometrically conforming, i.e., $\forall i \neq j$, the intersection $\partial \Omega_i \cap \partial \Omega_j$ is empty or is a common corner or edge of $\Omega_i$ and $\Omega_j$, where here and below an edge means a curve of continuous intervals and its two endpoints are called corners and the collection of these corners on $\partial \Omega_i$ are referred to corners of $\Omega_i$. Let us denote $F_{ij} := \partial \Omega_i \cap \partial \Omega_j$ as an edge of $\partial \Omega_i$ and $\bar{F}_{ij} := \partial \Omega_j \cap \partial \Omega_i$ an edge of $\partial \Omega_j$. In spite of the common edge $F_{ij}$ and $\bar{F}_{ij}$ being geometrically the same, they will be treated separately since we consider different triangulations on $F_{ij} \subset \partial \Omega_i$ with a mesh parameter $h_i$ and on $\bar{F}_{ij} \subset \partial \Omega_j$ with a mesh parameter $h_j$. Sometimes we use the notation $F_{ijh}$ and $\bar{F}_{ijh}$ to refer the sets of nodal points of the triangulation on $F_{ij}$ and $\bar{F}_{ij}$ with parameters $h_i$ and $h_j$, respectively, and $F_{ijh}$ and $\bar{F}_{ijh}$ when the endpoints are included. For preconditioning analysis, the above notation will be more convenient for us than using the usual notation $F_{ij}^h$ and $\bar{F}_{ij}^h$ sometimes found in the literature. We assume $f \in L^2(\Omega)$, and for simplicity of presentation let $\rho_i(x)$ be a positive constant $\rho_i$.

2.2. Discrete problem. Let us introduce a shape regular and quasiuniform triangulation with triangular elements in each $\Omega_i$ and let $h_i$ be the mesh size. The resulting triangulation on $\Omega$ is in general nonmatching across $\partial \Omega_i$. Let $X_i(\Omega_i)$ be the regular finite element (FE) space of piecewise linear and continuous functions in $\Omega_i$ and define

\begin{equation}
(2.2) \quad X(\Omega) = \prod_{i=1}^{N} X_i(\Omega_i) \equiv X_1(\Omega_1) \times X_2(\Omega_2) \times \cdots \times X_N(\Omega_N).
\end{equation}

We note that we do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial \Omega_i \cap \partial \Omega$.

Let us denote by $E^0_j$ the set of indices $j$ of $\Omega_j$ which has a common edge $F_{ij}$ with $\Omega_i$. To take into account also edges of $\Omega_i$ which belong to $\partial \Omega$, let us introduce a set of indices $E^0_i$ to refer these edges. The set of indices of all edges of $\Omega_i$ is denoted by $E_i := E^0_i \cup E^\partial_i$. A discrete problem obtained by a composite FE/DG method, see [31, 4, 11], is of the form: Find $u^* = \{u_i^*\}_{i=1}^{N} \in X(\Omega)$ where $u_i \in X_i(\Omega_i)$, such that

\begin{equation}
(2.3) \quad a_h(u^*, v) = f(v) \quad \forall v = \{v_i\}_{i=1}^{N} \in X(\Omega),
\end{equation}

where

\begin{equation}
(2.4) \quad a_h(u, v) := \sum_{i=1}^{N} a^i_h(u, v) \quad \text{and} \quad f(v) := \sum_{i=1}^{N} \int_{\Omega_i} f v_i \, dx,
\end{equation}

\begin{equation}
(2.5) \quad a^i_h(u, v) := a_i(u, v) + s_i(u, v) + p_i(u, v),
\end{equation}
and
\begin{equation}
    a_i(u, v) := \int_{\Omega_i} \rho_i \nabla u_i \cdot \nabla v_i \, dx,
\end{equation}

\begin{equation}
    s_i(u, v) := \sum_{j \in E_i} \int_{F_{ij}} \frac{1}{l_{ij}} \left( \rho_i \frac{\partial u_j}{\partial n} (v_j - v_i) + \rho_i \frac{\partial v_j}{\partial n} (u_j - u_i) \right) \, ds
\end{equation}

and
\begin{equation}
    p_i(u, v) := \sum_{j \in E_i} \int_{F_{ij}} \frac{\delta}{l_{ij}} \rho_i (u_j - u_i) (v_j - v_i) \, ds.
\end{equation}

Here, when \( j \in E_i^0 \) we set \( l_{ij} = 2 \) and let \( h_{ij} := 2 h_i h_j / (h_i + h_j) \), i.e., the harmonic average of \( h_i \) and \( h_j \). When \( j \in E_{i\partial} \) we denote the boundary edges \( F_{ij} \subset \partial \Omega_i \) by \( F_{i\partial} \) and set \( l_{i\partial} = 1 \) and \( h_{i\partial} = h_i \), and on the artificial edge \( F_{ji} \equiv F_{i\partial} \) we set \( u_{i\partial} = 0 \) and \( v_{i\partial} = 0 \). The partial derivative \( \frac{\partial}{\partial n} \) denotes the outward normal derivative on \( \partial \Omega_i \) and \( \delta \) is the penalty positive parameter.

We introduce the bilinear forms
\begin{equation}
    d_i(u, v) := a_i(u, v) + p_i(u, v)
\end{equation}

and
\begin{equation}
    d_h(u, v) := \sum_{i=1}^N d_i(u, v),
\end{equation}

and note that the norm defined by \( d_h(\cdot, \cdot) \) is a broken norm in \( X(\Omega) \) with weights given by \( \rho_i \) and \( \frac{\delta}{l_{ij} h_{ij}} \). For \( u = \{ u_i \}_{i=1}^N \in X(\Omega) \) this discrete norm is defined by
\[
\| u \|_h^2 := d_h(u, u) = \sum_{i=1}^N \left\{ \rho_i \| \nabla u_i \|_{L^2(\Omega_i)}^2 + \sum_{j \in E_i} \delta \frac{\rho_i}{l_{ij} h_{ij}} \int_{F_{ij}} (u_i - u_j)^2 \, ds \right\}.
\]

It is known that there exists a \( \delta_0 = O(1) > 0 \) and a positive constant \( c \), which does not depend on \( \rho_i, H_i, h_i \), and \( u \), such that for every \( \delta \geq \delta_0 \), we obtain \( |s_i(u, u)| \leq cd_i(u, u) \) and \( \sum_i |s_i(u, u)| \leq cd_h(u, u) \), and therefore, the following lemma is valid:

**Lemma 2.1.** There exists \( \delta_0 > 0 \) such that for \( \delta \geq \delta_0 \) and for all \( u \in X(\Omega) \) we have
\begin{equation}
    \gamma_0 d_i(u, u) \leq a_i'(u, u) \leq \gamma_1 d_i(u, u), \quad 1 \leq i \leq N
\end{equation}

and
\begin{equation}
    \gamma_0 d_h(u, u) \leq a_h(u, u) \leq \gamma_1 d_h(u, u).
\end{equation}

Here, \( \gamma_0 \) and \( \gamma_1 \) are positive constants independent of the \( \rho_i, h_i, H_i \) and \( u \). For the proof we refer to [11, 12].
Lemma 2.1 implies that the discrete problem (2.3) is elliptic and continuous, therefore, the solution exists and it is unique and stable. Lemma 2.1 together with Lemma 3.1, see below, are going to be fundamental for establishing condition number estimates for the FETI-DP preconditioned system developed in the remaining sections. An optimal a priori error estimate of this method was established in [3, 4] for the continuous coefficient case. When the coefficients are discontinuous across substructures and/or when \( h_i \) and \( h_j \) are not necessary of the same order, the following result is established in [11].

**Lemma 2.2.** Let \( u^*_{ex} \) and \( u^* \) be the solution of (2.1) and (2.3). If \( u^*_{ex}|_{\Omega_i} \in H^{1+s}(\Omega_i), 1 \leq i \leq N, \) and \( 1/2 \leq s \leq 1, \) then

\[
\| u^*_{ex} - u^* \|^2 \leq C \sum_{i=1}^{N} \left( h_i^{2s} + \sum_{j \in \mathcal{E}^0_i} \frac{h_j}{h_i} h_j^{2s} \right) \rho_i \| u^*_{ex} \|^2_{H^{1+s}(\Omega_i)}
\]

where \( C \) is independent of \( h_i, H_i, \rho_i, \) and \( u^*_{ex}. \)

We note that recently in [9], for matching meshes, it was established a quasi-optimal a priori error estimate whose solutions are only in \( H^{1+s} \) smooth with \( s \in (0, 1/2], \) i.e., covering the case in which less regularity of the solutions is assumed.

3. **Schur complement systems and discrete harmonic extensions.** The first step of many iterative substructuring solvers, such as the FETI-DP method that we consider in this paper, requires the elimination of unknowns interior to the subdomains. In this section, we describe this step for composite FE/DG discretizations.

![Illustration of the node classification on \( \Omega'_i. \)](image-url)

We introduce, see Figure 3.1, some notation and then formulate (2.3) as a variational problem with constraints. Let us introduce

\[
\Omega'_i := \overline{\Omega}_i \cup \{ \bigcup_{j \in \mathcal{E}^0_i} \overline{F}_{ji} \}
\]

i.e., the union of \( \overline{\Omega}_i \) and the \( \overline{F}_{ji} \subset \partial \Omega_j \) such that \( j \in \mathcal{E}^0_i, \) and let

\[
\Gamma_i := \partial \overline{\Omega}_i \setminus \partial \Omega \quad \text{and} \quad \Gamma'_i := \Gamma_i \cup \{ \bigcup_{j \in \mathcal{E}^0_i} \overline{F}_{ji} \}.
\]
We also introduce the sets

\[(3.1) \quad \Gamma := \bigcup_{i=1}^{N} \Gamma_i, \quad \Gamma' := \bigcap_{i=1}^{N} \Gamma'_i, \quad I_i := \Omega'_i \setminus \Gamma'_i \quad \text{and} \quad I := \bigcap_{i=1}^{N} I_i.\]

Let \(W_i(\Omega'_i)\) be the FE space of functions defined by the nodal values on \(\Omega'_i\)

\[(3.2) \quad W_i(\Omega'_i) = W_i(\Omega'_i) \times \prod_{j \in E'_i} W_i(\bar{F}_{ji}),\]

where \(W_i(\bar{F}_{ji})\) is the trace of the FE space \(X_j(\Omega_j)\) on \(\bar{F}_{ji}\) for all \(j \in E'_i\). A function \(u'_i \in W_i(\Omega'_i)\) is defined by the nodal values on \(\Omega'_i\), i.e., by the nodal values on \(\Omega_i\) and on \(\bar{F}_{ji}\) for all \(j \in E'_i\). Below, we will denote \(u'_i\) by \(u_i\) if it is not confused with functions of \(X_i(\Omega_i)\). A function \(u_i \in W_i(\Omega'_i)\) will be represented as

\[u_i = \{(u_i)_i, (u_i)_j \in E'_i\},\]

where \((u_i)_i := u_i|\Omega_i\) (\(u_i\) restricted to \(\Omega_i\)) and \((u_i)_j := u_i|\bar{F}_{ji}\) (\(u_i\) restricted to \(\bar{F}_{ji}\)).

Here and below we use the same notation to denote both FE functions and their vector representations. Note that \(a'_i(\cdot, \cdot)\), see (2.5), is defined on \(W_i(\Omega'_i) \times W_i(\Omega'_i)\) with corresponding stiffness matrix \(A'_i\) defined by

\[(3.3) \quad a'_i(u_i, v_i) = \langle A'_i u_i, v_i \rangle \quad u_i, v_i \in W_i(\Omega'_i),\]

where \(\langle u_i, v_i \rangle\) denotes the \(\ell_2\) inner product associated to nodes associated to the vector space in consideration. We also will represent \(u_i \in W_i(\Omega'_i)\) as \(u_i = (u_{i,I}, u_{i,I'})\) where \(u_{i,I'}\) represents values of \(u_i\) at nodal points on \(\Gamma'_i\) and \(u_{i,I}\) at the interior nodal points on \(I_i\), see (3.1). Hence, let us represent \(W_i(\Omega'_i)\) as the vector spaces \(W_i(I_i) \times W_i(\Gamma'_i)\).

Using the representation \(u_i = (u_{i,I}, u_{i,I'})\), the matrix \(A'_i\) can be represented as

\[(3.4) \quad A'_i = \begin{pmatrix} A'_{i,I,I} & A'_{i,I,I'} \\ A'_{i,I',I} & A'_{i,I',I'} \end{pmatrix},\]

where the block rows and columns correspond to the nodal points of \(I_i\) and \(\Gamma'_i\), respectively.

The Schur complement of \(A'_i\) with respect to \(u_{i,I'}\) is of the form:

\[(3.5) \quad S'_i := A'_{i,I',I'} - A'_{i,I'I}(A'_{i,I,I})^{-1}A'_{i,I'I}.\]

Note that \(S'_i\) satisfies

\[(3.6) \quad \langle S'_i u_{i,I'}, v_{i,I'} \rangle = \min_{w_{i,I'}} a'_{i}(w_{i}, w_{i}),\]

where the minimum is taken over \(w_{i} = (w_{i,I}, w_{i,I'}) \in W_i(\Omega'_i)\) such that \(w_{i,I'} = u_{i,I'}\) on \(\Gamma'_i\). The bilinear form \(a'_i(\cdot, \cdot)\) is symmetric and nonnegative with respect to \(W_i(\Omega'_i)\), see Lemma 2.1. The minimizing function satisfying (3.6) is called discrete harmonic in the sense of \(a'_i(\cdot, \cdot)\) or in the sense of \(\mathcal{H}'_i\). An equivalent definition of the minimizing function \(\mathcal{H}'_i u_{i,I'} \in W_i(\Omega'_i)\) is given by the solution of

\[(3.7) \quad a'_i(\mathcal{H}'_i u_{i,I'}, v_{i}) = 0 \quad v_{i} \in W_i(\Omega'_i).\]
Let us also introduce \( \mathcal{H}_i u_{i, \Gamma'} \in W_i(\Omega'_i) \) as the standard discrete harmonic function of \( u_{i, \Gamma'} \in W_i(\Gamma'_i) \), i.e., \( \mathcal{H}_i u_{i, \Gamma'} = u_{i, \Gamma'} \) on \( \Gamma'_i \) and discrete harmonic in \( \Omega_i \) in the sense of \( a_i(\cdot, \cdot) \), i.e.,

\[
a_i(\mathcal{H}_i u_{i, \Gamma'}, v_i) = 0 \quad v_i \in W_i(\Omega'_i)
\]

where we reinterpret \( a_i(\cdot, \cdot) \), see (2.6), as a bilinear form with respect to \( W_i(\Omega'_i) \). We note that the extensions \( \mathcal{H}_i \) and \( \mathcal{H}_i' \) differ from each other since \( \mathcal{H}_i u_{i, \Gamma'} \) at the interior nodes \( I_i \) depends only on the nodal values of \( u_{i, \Gamma'} \) on \( \Gamma_i \), while \( \mathcal{H}_i' u_{i, \Gamma'} \) depends on the nodal values of \( u_{i, \Gamma'} \) on \( \Gamma'_i \). The following lemma shows the equivalence (in the energy form defined by \( d_i(\cdot, \cdot) \)) between discrete harmonic functions in the sense of \( \mathcal{H}_i \) and in the sense of \( \mathcal{H}_i' \); for the proof see Lemma 4.1 of [12]. This equivalence allows us to take advantages of all the discrete Sobolev results known for \( \mathcal{H}_i \) discrete harmonic extensions.

**Lemma 3.1.** For \( u_{i, \Gamma'} \in W_i(\Gamma'_i) \) it holds that

\[
d_i(\mathcal{H}_i u_{i, \Gamma'}, \mathcal{H}_i' u_{i, \Gamma'}) \leq d_i(\mathcal{H}_i' u_{i, \Gamma'}, \mathcal{H}_i' u_{i, \Gamma'}) \leq C d_i(\mathcal{H}_i u_{i, \Gamma'}, \mathcal{H}_i u_{i, \Gamma'})
\]

where \( C \) is a positive constant independent of \( h_i, H_i, \rho_i \) and \( u_{i, \Gamma'} \).

The next corollary follows directly from Lemma 3.1 and Lemma 2.1.

**Corollary 3.2.** For \( u_{i, \Gamma'} \in W_i(\Gamma'_i) \) it holds that

\[
C_0 d_i(\mathcal{H}_i u_{i, \Gamma'}, \mathcal{H}_i u_{i, \Gamma'}) \leq a_i(\mathcal{H}_i' u_{i, \Gamma'}, \mathcal{H}_i' u_{i, \Gamma'}) \leq C_1 d_i(\mathcal{H}_i u_{i, \Gamma'}, \mathcal{H}_i u_{i, \Gamma'})
\]

where \( C_0 \) and \( C_1 \) are positive constants independent of \( h_i, H_i, \rho_i \) and \( u_{i, \Gamma'} \).

Let us introduce the product space

\[
W(\Omega') := \prod_{i=1}^N W_i(\Omega'_i),
\]

i.e., \( u \in W(\Omega') \) means that \( u = \{u_i\}_{i=1}^N \) where \( u_i \in W_i(\Omega'_i) \); see (3.2) for the definition of \( W_i(\Omega'_i) \). We remind that \( (u_i)_{i,j} = u_i|_{\Omega_i} \) (\( u_i \) restricted to \( \Omega_i \)) and \( (u_i)_{j,i} = u_i|_{\Gamma_j} \) (\( u_i \) restricted to \( \Gamma_j \)). Using the representation \( u_i = (u_{i,t}, u_{i,r}) \) where \( u_{i,t} \in W_i(I_i) \) and \( u_{i,r} \in W_i(\Gamma'_i) \), see (3.4), let us introduce the product space

\[
W(\Gamma') := \prod_{i=1}^N W_i(\Gamma'_i),
\]

i.e., \( u_{i,r} \in W(\Gamma') \) means that \( u_{i,r} = \{u_{i,r,i}\}_{i=1}^N \) where \( u_{i,r,i} \in W_i(\Gamma'_i) \). The space \( W(\Gamma') \) which was defined on \( \Gamma' \) only, also will be interpreted below as the subspace of \( W(\Omega') \).
of functions which are discrete harmonic in the sense of $H'_i$ in each $\Omega_i$.

We now consider the subspace $\hat{W}(\Omega') \subset W(\Omega')$ and $\hat{W}(\Gamma') \subset W(\Gamma')$ as the space of functions which are continuous on $\Gamma$ in the sense of the following definition (for notation see (3.1)):

\[
\text{(Spaces } \hat{W}(\Omega') \text{ and } \hat{W}(\Gamma').) \text{ We say that } u = \{u_i\}_{i=1}^{N} \in W(\Omega') \text{ is continuous on } \Gamma \text{ if for all } 1 \leq i \leq N \text{ we have }
\]

\[
(u_i)_i(x) = (u_j)_i(x) \quad \text{for all } x \in \bar{F}_{ij} \text{ for all } j \in E_0^i \quad \text{(3.13)}
\]

and

\[
(u_i)_j(x) = (u_j)_j(x) \quad \text{for all } x \in \bar{F}_{ji} \text{ for all } j \in E_0^i. \quad \text{(3.14)}
\]

In Figure 3.2 we illustrate this continuity by assigning the same nodal value for nodes connected by a line. The subspace of $W(\Omega')$ of continuous functions on $\Gamma$ is denoted by $\hat{W}(\Omega')$, and the subspace of $W(\Omega')$ of functions which are discrete harmonic in the sense of $H'_i$ in each $\Omega_i$ is denoted by $\hat{W}(\Gamma')$.

Note that there is a one-to-one correspondence between vectors in the spaces $X(\Omega)$ and $W(\Omega')$. Indeed, let us introduce the restriction matrices $R_{Ω'_i} : X(\Omega) \to W_i(Ω'_i)$ which assign uniquely the vector values of $u = \{u_i\}_{i=1}^{N} \in X(\Omega)$ where $u_i \in X_i(Ω_i)$, see (2.2), to $v_i \in W_i(Ω'_i)$ defined by $(v_i)_i = u_i$ on $Ω_i$ and $(v_i)_j = u_j$ on $F_{ij}$ for all $j \in E_0^i$. It is easy to see that $v := \{v_i\}_{i=1}^{N}$ where $v_i := R_{Ω'_i}u$ belongs to $W(Ω)$. And vice-versa, for each $v = \{v_i\}_{i=1}^{N} \in W(Ω)$, we can define uniquely $u = \{u_i\}_{i=1}^{N} \in X(\Omega)$ by setting $u_i = (v_i)_i$. Since $u$ and $v$ have identical nodal values, we will refer sometimes $u \in W(Ω')$ or $u \in X(Ω)$. For instance, the solution $u^*$ of (2.3) can be interpreted as a function in $W(Ω')$ or in $X(Ω)$.

Note that the discrete problem (2.3) can be written as a system of algebraic equations

\[
\hat{A}u^* = f \quad \text{(3.15)}
\]
for \( u^* \in X(\Omega) \) using the standard FE basis functions, and \( f = \{ f_i \}_{i=1}^N \in X(\Omega) \), where \( f_i \) is the load vector associated with individual subdomains \( \Omega_i \), i.e., is \( \int_{\Omega_i} f v_i \) when \( v_i \) are the canonical basis functions of \( X_i(\Omega_i) \). The stiffness matrix \( \hat{A} \) can be obtained by assembling the matrices \( A_i' \), see (3.3), from \( W(\Omega') \) to \( X(\Omega) \) as

\[
\hat{A} = \sum_{i=1}^N R_{i,\Omega}^T A_i' R_{i,\Omega'}.
\]

Note that the matrix \( \hat{A} \) is no longer block diagonal since there are couplings between substructures due to the continuity on \( \Gamma \), see Definition 3.3.

Note also that \( X(\Omega) \) can be componentwise represented by \( X(I) \times X(\Gamma) \), denoted also by \( X(\Omega) \), where \( X(I) := \prod_{i=1}^N X_i(I_i) \) is the vector space of functions defined by nodal values on \( I_i \), and \( X(\Gamma) := \prod_{i=1}^N X_i(\Gamma_i) \) by the nodal values on \( \Gamma_i \), see (3.1) and Figure 3.1. Hence, we can represent \( u \in X(\Omega) \) as \( u = (u_I, u_\Gamma) \) with \( u_I \in X(I) \) and \( u_\Gamma \in X(\Gamma) \). We introduce the restriction \( R_{\Gamma_i} : X(\Gamma) \to W_i(\Gamma'_i) \) by assigning values \( u_\Gamma \in X(\Gamma) \) into \( u_i \in W_i(\Gamma'_i) \) at the nodes of \( \Gamma'_i \). By eliminating the variable \( u_I = \{ u_{i,I} \}_{i=1}^N \) of \( u^* = (u_I, u_\Gamma) \) from (3.15), see (3.4) and (3.5), it is easy to see that

\[
(3.16) \quad \hat{S} u_{\Gamma}^* = \hat{g}_{\Gamma}
\]

where

\[
(3.17) \quad \hat{S} = \sum_{i=1}^N R_{i,\Omega}^T S_i' R_{i,\Omega'} \quad \text{and} \quad \hat{g}_{\Gamma} = f_\Gamma - \sum_{i=1}^N R_{i,\Omega}^T A_i' R_{i,\Omega'} (A_i',\Gamma')^{-1} f_i,
\]

where \( f_\Gamma := \{ f_i,\Gamma \}_{i=1}^N \) with \( f_i,\Gamma \in X_i(\Gamma_i) \), where the load vector \( f_i,\Gamma \) is defined by \( \int_{\Omega} f v_i,\Gamma \) when \( v_i,\Gamma \) are the canonical basis functions of \( X_i(\Omega_i) \) associated to nodes on \( \Gamma_i \). It is also easy to see from (3.7) and (3.8) that

\[
(3.18) \quad \begin{pmatrix} v_{i,I} \\ v_{i,\Gamma} \end{pmatrix}^T \begin{pmatrix} A_i',\Gamma' \\ A_i',\Gamma' \end{pmatrix} \begin{pmatrix} \mathcal{H}_i' u_{i,\Gamma'} \\ u_{i,\Gamma'} \end{pmatrix} = \langle S_i' u_{i,\Gamma'}, v_{i,\Gamma'} \rangle.
\]

Note that \( \hat{W}(\Gamma') \) is the natural space for defining \( \langle \hat{S} \cdot, \cdot \rangle \) due to (3.17), (3.18) and the continuity of \( \hat{W}(\Gamma') \) on \( \Gamma \), see Definition 3.3.

4. FETI-DP with corner constraints. We now design a FETI-DP method for solving (3.16). We follow the abstract approach described in pages 160-167 in [34].

For \( 1 \leq i \leq N \), we introduce the nodal points associated to the corner unknowns, see Figure 3.1, by

\[
(4.1) \quad V_i' := V_i \cup \{ \cup_{j \in E_i^0} \partial F_{ij} \} \quad \text{where} \quad V_i := \{ \cup_{j \in E_i^0} \partial F_{ij} \}.
\]

We now consider the subspace \( \hat{W}(\Omega') \subset W(\Omega') \) and \( \hat{W}(\Gamma') \subset W(\Gamma') \) as the space of functions which are continuous on all the \( V_i \) in the sense of the following definition:

**Definition 4.1.** (Subspaces \( \hat{W}(\Omega') \) and \( \hat{W}(\Gamma') \)). We say that \( u = \{ u_i \}_{i=1}^N \in W(\Omega') \) is continuous at the corners \( V_i' \) if for \( 1 \leq i \leq N \) we have

\[
(4.2) \quad (u_i)_i(x) = (u_j)_j(x) \quad \text{at} \ x \in \partial F_{ij} \quad \text{for all} \ j \in E_i^0
\]
and

\[(u_i)_j(x) = (u_j)_j(x) \quad \text{at} \quad x \in \partial F_{ji} \quad \text{for all} \quad j \in \mathcal{E}_i^0.\]

In Figure 4.1 we illustrate this continuity by assigning the same nodal value for nodes (corners) connected by a line. The subspace of \(W(\Omega')\) of continuous functions at the corners \(V_i'\) for all \(1 \leq i \leq N\) is denoted by \(\tilde{W}(\Omega')\), and the subspace of \(\tilde{W}(\Omega')\) of functions which are discrete harmonic in the sense of \(\mathcal{H}_i'\) is denoted by \(\tilde{W}(\Gamma')\).

Note that

\[\tilde{W}(\Gamma') \subset \tilde{W}(\Gamma') \subset W(\Gamma').\]

Let \(\tilde{A}\) be the stiffness matrix which is obtained by assembling the matrices \(A'_i\) for \(1 \leq i \leq N\), from \(W(\Omega')\) to \(W(\Omega')\). Note that the matrix \(\tilde{A}\) is no longer block diagonal since there are couplings between variables at the corners \(V_i'\) for \(1 \leq i \leq N\). We will represent \(u \in \tilde{W}(\Omega')\) as \(u = (u_I, u_{\Pi}, u_{\Delta})\) where the subscript \(I\) refers to the interior degrees of freedom at nodal points \(I = \prod_{i=1}^N I_i\), the \(\Pi\) refers to the corners \(V_i'\) for all \(1 \leq i \leq N\), and the \(\Delta\) refers to the remaining nodal points, i.e., the nodal points \(\Gamma_i' \backslash V_i'\) for all \(1 \leq i \leq N\). The vector \(u = (u_I, u_{\Pi}, u_{\Delta}) \in \tilde{W}(\Omega')\) is obtained from the vector \(u = \{u_i\}_{i=1}^N \in W(\Omega')\) using the equations (4.2), (4.3), i.e., the continuity of \(u\) on \(V_i'\) for all \(1 \leq i \leq N\). Using the decomposition \(u = (u_I, u_{\Pi}, u_{\Delta}) \in \tilde{W}(\Omega')\) we can partition \(\tilde{A}\) as

\[
\tilde{A} = \begin{pmatrix}
A'_{II} & A'_{I\Pi} & A'_{I\Delta} \\
A'_{\Pi I} & A'_{\Pi\Pi} & A'_{\Pi\Delta} \\
A'_{\Delta I} & A'_{\Delta\Pi} & A'_{\Delta\Delta}
\end{pmatrix}.
\]

We note that the only couplings across subdomains are through the variables \(\Pi\) where the matrix \(\tilde{A}\) is subassembled.
A Schur complement of $\hat{A}$ with respect to the $\triangle$-unknowns (eliminating the $I$- and the $II$-unknowns) is of the form

$$\hat{S} := A'_{\triangle\triangle} - (A'_{\triangle I} A'_{I I}) \left( A'_{I I} \hat{A}_{I I} A'_{I I} \right)^{-1} \left( A'_{I I} \hat{A}_{I I} A'_{\triangle \triangle} \right),$$

where

$$\hat{S} = \min \{ \hat{S} u_{\triangle}, u_{\triangle} \} \quad \text{where the minimum is taken over } w = (w_I, w_{\Pi}, w_{\triangle}) \in \hat{W}(\Omega) \text{ such that } w_{\triangle} = u_{\triangle}. \quad (4.6)$$

A vector $u \in \hat{W}(\Gamma')$ can uniquely be represented by $u = (u_{\Pi}, u_{\triangle})$, therefore, we can represent $W(\Gamma') = \hat{W}_{\Pi}(\Gamma') \times W_{\triangle}(\Gamma')$, where $\hat{W}_{\Pi}(\Gamma')$ refers to the $II$-degrees of freedom of $W(\Gamma')$ while $W_{\triangle}(\Gamma')$ to the $\triangle$-degrees of freedom of $\hat{W}(\Gamma')$. The vector space $W_{\triangle}(\Gamma')$ can be decomposed as

$$W_{\triangle}(\Gamma') = \bigoplus_{i=1}^{N} W_{i,\triangle}(\Gamma'_i) \quad (4.7)$$

where the local space $W_{i,\triangle}(\Gamma'_i)$ refers to the degrees of freedom associated to the nodes of $\Gamma'_i \setminus \Omega'_i$ for $i = 1, \ldots, N$. Hence, a vector $u \in \hat{W}(\Gamma')$ can be represented as $u = (u_{\Pi}, u_{\triangle})$ with $u_{\Pi} \in \hat{W}_{\Pi}(\Gamma')$ and $u_{\triangle} = \{u_{i,\triangle}\}_{i=1}^{N} \in W_{\triangle}(\Gamma')$ where $u_{i,\triangle} \in W_{i,\triangle}(\Gamma'_i)$. Note that $\hat{S}$, see (4.6), is defined on the vector space $W_{\triangle}(\Gamma')$, and the following lemma follows (cf. Lemma 6.22 in [34] and Lemma 4.2 in [30]):

**Lemma 4.2.** Let $u_{\triangle} \in W_{\triangle}(\Gamma')$ and let $\hat{S}$ and $\hat{A}$, defined in (4.6) and (4.5), respectively. Then,

$$\langle \hat{S} u_{\triangle}, u_{\triangle} \rangle = \min \{ \hat{A} w, w \} \quad (4.8)$$

where the minimum is taken over $w = (w_I, w_{\Pi}, w_{\triangle}) \in \hat{W}(\Omega)$ such that $w_{\triangle} = u_{\triangle}$.

Let us take $u \in \hat{W}(\Gamma')$ as $u = (u_{\Pi}, u_{\triangle})$ with $u_{\Pi} \in \hat{W}_{\Pi}(\Gamma')$ and $u_{\triangle} \in W_{\triangle}(\Gamma')$, where $u_{\triangle} = \{u_{i,\triangle}\}_{i=1}^{N}$ with $u_{i,\triangle} \in W_{i,\triangle}(\Gamma'_i)$. The vector $u_{i,\triangle} \in W_{i,\triangle}(\Gamma'_i)$ can be partitioned as

$$u_{i,\triangle} = \{(u_{i,\triangle}), \{(u_{i,\triangle})_j\}_{j \in \mathcal{E}^0}\}$$

where

$$(u_{i,\triangle})_i = u_{i,\triangle}|_{\Gamma'_i \setminus \Omega'_i} \text{ and } (u_{i,\triangle})_j = u_{i,\triangle}|_{F_{ji}}.$$  

In order to measure the jump of $u_{\triangle} \in W_{\triangle}(\Gamma')$ across the $\triangle$-nodes let us introduce the space $\hat{W}_{\triangle}(\Gamma)$ defined by

$$\hat{W}_{\triangle}(\Gamma) = \bigoplus_{i=1}^{N} X_i(\Gamma_i \setminus \Omega_i),$$

where $X_i(\Gamma_i \setminus \Omega_i)$ is the restriction of $X_i(\Omega_i)$ to $\Gamma_i \setminus \Omega_i$, see Figure 3.1. To define the jumping matrix $B_{\triangle} : W_{\triangle}(\Gamma') \rightarrow \hat{W}_{\triangle}(\Gamma)$, let $u_{\triangle} = \{u_{i,\triangle}\}_{i=1}^{N} \in W_{\triangle}(\Gamma')$ and let $v := B_{\triangle} u$ where $v = \{v_i\}_{i=1}^{N} \in \hat{W}_{\triangle}(\Gamma)$ is defined by

$$v_i = (u_{i,\triangle})_i - (u_{j,\triangle})_i \quad \text{on } F_{ij} \quad \text{for all } j \in \mathcal{E}^0_i. \quad (4.9)$$

The jumping matrix $B_{\triangle}$ can be written as

$$B_{\triangle} = (B_{\triangle}^{(1)}, B_{\triangle}^{(2)}, \ldots, B_{\triangle}^{(N)}). \quad (4.10)$$
where the rectangular matrix $B_{\Delta}^{(i)}$ consists of columns of $B_{\Delta}$ attributed to the $(i)$ components of functions of $W_{i,\Delta}(\Gamma')$ of the product space $W_{\Delta}(\Gamma')$, see (4.7). The entries of the rectangular matrix consist of values of $\{0, 1, -1\}$. It is easy to see that the Range $B_{\Delta} = W_{\Delta}(\Gamma')$, $B_{\Delta}$ is full rank. In addition, if $u = (u_{II}, u_{\Delta}) \in \tilde{W}(\Gamma')$ and $B_{\Delta}u_{\Delta} = 0$ then $u \in \tilde{W}(\Gamma')$.

We can reformulate the problem (3.16) as the variational problem with constraints in $W_{\Delta}(\Gamma')$ space: Find $u_{\Delta}^* \in W_{\Delta}(\Gamma')$ such that
\begin{equation}
J(u_{\Delta}^*) = \min J(v_{\Delta})
\end{equation}
subject to $v_{\Delta} \in W_{\Delta}(\Gamma')$ with constraints $B_{\Delta}v_{\Delta} = 0$, where
\begin{equation}
J(v_{\Delta}) := 1/2\langle \tilde{S}v_{\Delta}, v_{\Delta} \rangle - \langle \tilde{g}_{\Delta}, v_{\Delta} \rangle
\end{equation}
where $\tilde{S}$ is defined in (4.6) and
\begin{equation}
\tilde{g}_{\Delta} := f_{\Delta} - (A_{\Delta I} A_{\Delta II})(A_{II}^{I} A_{III}^{I})^{-1} \left( f_{I} A_{II}^{I} f_{II}^{I} \right).
\end{equation}
We note that $f = \{f_i\}_{i=1}^{N} \in X(\Omega)$ was defined in (3.15) and it can be represented as $f = (f_{I}, f_{II}, f_{I,\Delta})$. It remains to define $f_{\Delta}$ in (4.13). The forcing term $f_{\Delta} \in W_{\Delta}(\Gamma')$ is defined by $f_{\Delta} = \{f_{i,\Delta}\}_{i=1}^{N}$ where $f_{i,\Delta}$ are the load vectors associated with the individual subdomains $\Omega_{i}$, i.e., the entries $f_{i,\Delta}$ are defined as $\int_{\Omega_{i}} f_{i,\Delta} dx$ when $v_{i,\Delta}$ are the canonical basis functions of $W_{i,\Delta}(\Gamma')$. Note that $\tilde{S}$ is symmetric and positive definite since $A$ has these properties; see also Lemma 4.2. Introducing Lagrange multipliers $\lambda \in \tilde{W}_{\Delta}(\Gamma)$, the problem (4.11) reduces to the saddle point problem of the form: Find $u_{\Delta}^* \in W_{\Delta}(\Gamma')$ and $\lambda^* \in \tilde{W}_{\Delta}(\Gamma)$ such that
\begin{equation}
\begin{cases}
\tilde{S}u_{\Delta} + B_{\Delta}^T \lambda^* = \tilde{g}_{\Delta} \\
B_{\Delta}u_{\Delta}^* = 0.
\end{cases}
\end{equation}
Hence, (4.14) reduces to
\begin{equation}
FX^* = g
\end{equation}
where
\begin{equation}
F := B_{\Delta} \tilde{S}^{-1} B_{\Delta}^T, \quad g := B_{\Delta} \tilde{S}^{-1} \tilde{g}_{\Delta}.
\end{equation}
When $\lambda^*$ is computed, $u_{\Delta}^*$ can be found by solving the problem
\begin{equation}
\tilde{S}u_{\Delta}^* = \tilde{g}_{\Delta} - B_{\Delta}^T \lambda^*.
\end{equation}

### 4.1. Dirichlet Preconditioner

We now define the FETI-DP preconditioner for $F$, see (4.16). Let $S'_{i,\Delta}$ be the Schur complement of $S_{i}$, see (3.5), restricted to $W_{i,\Delta}(\Gamma') \subset W_{i}(\Gamma')$, i.e., taken $S_{i}'$ on functions in $W_{i}(\Gamma')$ which vanish on $\mathcal{V}_{i}$, Let
\begin{equation}
S_{\Delta}' := \text{diag}(S_{i,\Delta}')_{i=1}^{N}.
\end{equation}
In other words, $S'_{i,\Delta}$ is obtained from $S'_i$ by deleting rows and columns corresponding to nodal values at nodal points of $V'_i \subset \Gamma'_i$.

Let us introduce diagonal scaling matrices $D^{(i)}_{\Delta} : W_{i,\Delta}(\Gamma'_i) \to W_{i,\Delta}(\Gamma'_i)$, for $1 \leq i \leq N$. The diagonal entry of $D^{(i)}_{\Delta}$ associated to a node $x \in \Gamma'_i \setminus V'_i$, which we denote by $D^{(i)}_{\Delta}(x)$, is defined by

$$D^{(i)}_{\Delta}(x) = \frac{\rho^\beta_i}{\rho^\beta_i + \rho^\beta_j}$$

for $\beta \in [1/2, \infty)$, see [32], and define

$$B_{D,\Delta} = (B^{(1)}_{\Delta} D^{(1)}_{\Delta}, \cdots, B^{(N)}_{\Delta} D^{(N)}_{\Delta}).$$

Let

$$P_{\Delta} := B^T_{D,\Delta} B_{\Delta}$$

which maps $W_{\Delta}(\Gamma')$ into itself. It is easy to check that for $w_{\Delta} = \{w_{i,\Delta}\}_{i=1}^N \in W_{\Delta}(\Gamma')$ and $v_{\Delta} := P_{\Delta} w_{\Delta}$, the following equalities hold:

$$(v_{i,\Delta})_i(x) = \frac{\rho^\beta_i}{\rho^\beta_i + \rho^\beta_j}[(w_{i,\Delta})_i(x) - (w_{j,\Delta})_i(x)], \quad x \in F_{ijh},$$

$$(v_{j,\Delta})_j(x) = \frac{\rho^\beta_j}{\rho^\beta_i + \rho^\beta_j}[(w_{i,\Delta})_j(x) - (w_{j,\Delta})_j(x)], \quad x \in F_{ijh}.$$

Note that

$$(v_{j,\Delta})_i(x) = \frac{\rho^\beta_i}{\rho^\beta_i + \rho^\beta_j}[(w_{j,\Delta})_i(x) - (w_{i,\Delta})_i(x)], \quad x \in F_{ijh},$$

$$(v_{j,\Delta})_j(x) = \frac{\rho^\beta_j}{\rho^\beta_i + \rho^\beta_j}[(w_{j,\Delta})_j(x) - (w_{i,\Delta})_j(x)], \quad x \in F_{ijh}.$$
Note that $M^{-1}$ is a block-diagonal matrix, and each block is invertible since $S'_{i,\Delta}$ and $D^{(i)}_{\Delta}$ are invertible and $B^{(i)}_{\Delta}$ is a full rank matrix. The following theorem holds:

**Theorem 4.3.** For any $\lambda \in \hat{W}_{\Delta}(\Gamma')$ it holds that

\begin{equation}
\langle M\lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq C(1 + \log \frac{H}{h})^2 \langle M\lambda, \lambda \rangle
\end{equation}

where $C$ is a positive constant independent of $h_i$, $h_i/h_j$, $H_i$, $\lambda$ and the jumps of $\rho_i$. Here and below, $\log(\frac{H}{h}) := \max_{i=1}^{N} \log(\frac{H}{h_i})$.

**Proof.** We follow to the general abstract theory for FETI-DP methods developed in Theorem 6.35 of [34]. This abstract theory relies only on duality and linear algebra arguments, and properties such as that $B_{\Delta}$ is full rank, $P_{\Delta}$ is a projection, $\hat{S}$ is invertible and the subspace inclusion (4.4). Using the same abstract arguments, the proof of the theorem follows by checking the Lemma 4.4 and Lemma 4.5, see below. The proofs of these two lemmas are not algebraic and they are dependent of the problem. The proofs of these two lemmas are presented separately below. \[\blacksquare\]

**Lemma 4.4.** For $u_{\Delta} \in W_{\Delta}(\Gamma')$ it follows that

\begin{equation}
\langle \hat{S}u_{\Delta}, u_{\Delta} \rangle \leq \langle S'_{\Delta}u_{\Delta}, u_{\Delta} \rangle.
\end{equation}

**Proof.** The proof follows from Lemma 4.2 and from

\begin{equation}
\langle \hat{S}u_{\Delta}, u_{\Delta} \rangle = \min \langle \hat{A}w, w \rangle \leq \min \langle \hat{A}v, v \rangle = \langle S'_{\Delta}u_{\Delta}, u_{\Delta} \rangle
\end{equation}

where the minima are taken over $w = (w_I, w_{II}, w_{\Delta}) \in \hat{W}(\Omega')$ such that $w_{\Delta} = u_{\Delta}$, and $v = (v_I, v_{II}, v_{\Delta}) \in \hat{W}(\Omega')$ such that $v_{II} = 0$ and $v_{\Delta} = u_{\Delta}$. \[\blacksquare\]

**Lemma 4.5.** For any $u_{\Delta} \in W_{\Delta}(\Gamma')$ it holds that

\begin{equation}
\| P_{\Delta}u_{\Delta} \|_{S_{\Delta}}^2 \leq C(1 + \log \frac{H}{h})^2 \| u_{\Delta} \|_{S}^2
\end{equation}

where $C$ is a positive constant independent of $h_i$, $h_i/h_j$, $H_i$, $u_{\Delta}$ and the jumps of $\rho_i$.

**Proof.** We first consider the case when the edges $F_{ij}$ are a single interval only. Let $u_{\Delta} \in W_{\Delta}(\Gamma')$ and let $u = (u_{II}, u_{\Delta}) \in \hat{W}(\Gamma')$ be the solution of

\begin{equation}
\langle \hat{S}u_{\Delta}, u_{\Delta} \rangle = \min \langle S'w, w \rangle = \langle S'u, u \rangle,
\end{equation}

where the minimum is taken over $w = (w_{II}, w_{\Delta}) \in \hat{W}(\Gamma')$ such that $w_{II} \in \hat{W}_{II}(\Gamma')$ and $w_{\Delta} = u_{\Delta}$. Hence, we can replace $\| u_{\Delta} \|_{S}$ in (4.31) by $\| u \|_{S'}$.

Let us represent the $u$ defined above as $\{u_i\}_{i=1}^{N} \in \hat{W}(\Gamma')$ where $u_i \in \hat{W}_i(\Gamma_i)$. Let $I_{F_{ij}}(u_i)$ be the linear function on $\hat{F}_{ij}$ defined by the values of $(u_i)_j$ at $x \in \partial F_{ij}$, and let $I_{F_{ji}}(u_j)$ be the linear function on $\hat{F}_{ji}$ defined by the values of $(u_j)_i$ at $x \in \partial F_{ji}$. Let $\hat{u} := \{\hat{u}_i\}_{i=1}^{N}$ where $\hat{u}_i \in \hat{W}_i(\Gamma'_i)$ is defined by

\begin{equation}
(\hat{u}_i)_j = I_{F_{ij}}(u_i)_j \text{ on } \hat{F}_{ijh} \text{ for all } j \in \mathcal{E}^0_i
\end{equation}
and 

$$
(\hat{u}_i)_j = I_{F_{ji}}(u_i)_j \text{ on } F_{ji}h \text{ for all } j \in E^0_i.
$$

Note that $\hat{u} \in \hat{W}(\Gamma')$, therefore, let us represent $\hat{u} = (\hat{u}_\Pi, \hat{u}_\Delta)$ where $B_\Delta \hat{u}_\Delta = 0$. Using this we have, see (4.21),

$$
P_\Delta u_\Delta \equiv B^T_{D_i,\Delta} B_{D_i} u_\Delta = B^T_{D_i,\Delta} B_{D_i}(u_\Delta - \hat{u}_\Delta) = P_\Delta(u_\Delta - \hat{u}_\Delta).
$$

Note that $u - \hat{u} = 0$ at the $\Pi$-nodes, hence, let us define $v \in W(\Gamma')$ to be equal to $P_\Delta(u_\Delta - \hat{u}_\Delta)$ at the $\Delta$-nodes and equal to zero at the $\Pi$-nodes. Let us represent $v = \{v_i\}_{i=1}^N$ where $v_i \in W_i(\Gamma'_i')$. We have

$$
\|P_\Delta u_\Delta\|_{S_\Delta'}^2 = \sum_{i=1}^N \|v_i\|_{S_i'}^2 \tag{4.33}
$$

in view of the definition of $S_{i,\Delta}'$ and $S_\Delta'$, see (4.18), (3.5) and (4.6), hence, to prove the lemma it remains to show that

$$
\sum_{i=1}^N \|v_i\|_{S_i'}^2 \leq C(1 + \log \frac{H}{h})^2 \|u\|_{S}^2 \tag{4.34}
$$

since by (4.32) we obtain (4.31). By Corollary 3.2 we need to show

$$
\sum_{i=1}^N \tilde{d}_i(v_i, v_i) \leq C(1 + \log \frac{H}{h})^2 \sum_{i=1}^N \tilde{d}_i(u_i, u_i) \tag{4.35}
$$

where, see (2.9),

$$
\tilde{d}_i(v_i, v_i) := d_i(\mathcal{H}_i v_i, \mathcal{H}_i v_i)
$$

and so

$$
\tilde{d}_i(v_i, v_i) = \rho_i \| \nabla(v_i) \|_{L^2(\Omega_i)}^2 + \sum_{j \in E_i} \frac{\rho_i \delta}{l_{ij}h_{ij}} \| (v_i)_i - (v_i)_j \|_{L^2(F_{ij})}^2, \tag{4.36}
$$

where $(v_i)_i = \mathcal{H}_i v_i$ and $(u_i)_i = \mathcal{H}_i u_i$ inside of the subdomains $\Omega_i$.

We first estimate the first term of (4.36). We have

$$
\| \nabla(v_i) \|_{L^2(\Omega_i)}^2 \leq C \sum_{j \in E_i^0} \| (v_i)_i \|_{H^{1/2}_{00}(F_{ij})}^2 \tag{4.37}
$$

by the well-known estimate, see [34], and the fact that $(v_i)_i = 0$ at corners of $\partial \Omega_i$. Note that (4.37) is also valid for subdomains $\Omega_i$ which intersect $\partial \Omega$ by edges since we use the obvious inequality

$$
\| \nabla(v_i) \|_{L^2(\Omega_i)}^2 \leq \| \nabla(\bar{v}_i) \|_{L^2(\Omega_i)}^2 \leq C \sum_{j \in E_i^0} \| (v_i)_i \|_{H^{1/2}_{00}(F_{ij})}^2
$$
where \((\tilde{v}_i)_i\) is the standard discrete harmonic extension on \(\Omega_i\) with \((\tilde{v}_i)_i = (v_i)_i\) on edges \(F_{ij}\) for \(j \in \mathcal{E}^0_i\), and \((\tilde{v}_i)_i = 0\) on edges \(F_{ij}\) for \(j \in \mathcal{E}^1_i\). For the case \(F_{ij}\) such that \(j \in \mathcal{E}^0_i\), we use (4.22) to get

\[
\rho_i \| (u_i)_i \|^2_{H^{1/2}(F_{ij})} = \frac{\rho_i \rho_j^2 \rho^2}{(\rho_i^2 + \rho_j^2)^2} \| (u_i - \tilde{u}_i)_i - (u_j - \tilde{u}_j)_j \|^2_{H^{1/2}(F_{ij})} \leq \\
3 \{ \rho_i \| (u_i - \tilde{u}_i)_i \|^2_{H^{1/2}(F_{ij})} + \rho_j \| (u_j - \tilde{u}_j)_j \|^2_{H^{1/2}(F_{ij})} + \\
+ \frac{\rho_i \rho_j^2 \rho^2}{(\rho_i^2 + \rho_j^2)^2} \| (u_j - \tilde{u}_j)_j - (u_j - \tilde{u}_j)_j \|^2_{H^{1/2}(F_{ij})} \} ,
\]

(4.38)

where we have used that \(\frac{\rho_i \rho_j^2 \rho^2}{(\rho_i^2 + \rho_j^2)^2} \leq \min\{\rho_i, \rho_j\} \) if \(\beta \in [1/2, \infty)\), see [32].

For estimating the first term of the right-hand side of (4.38), it is well-known that, see [34],

\[
\rho_i \| (u_i - \tilde{u}_i)_i \|^2_{H^{1/2}(F_{ij})} \leq C(1 + \log \frac{H_i}{h_i})^2 \rho_i \| \nabla (u_i)_i \|_{L^2(\Omega_i)} \leq \\
C(1 + \log \frac{H_i}{h_i})^2 d_i(u_i, u_i)
\]

since \((\tilde{u}_i)_i = I_{F_{ij}}(u_i)_i\) is the linear interpolant of \((u_i)_i\) on \(F_{ij}\) with nodal values \((u_i)_i\) given on \(\partial F_{ij}\). The estimate for the second term of the right-hand side of (4.38) is similar.

It remains to estimate the third term of the right-hand side of (4.38). We have

\[
\| (u_j - \tilde{u}_j)_j \|^2_{H^{1/2}(F_{ij})} = \\
= |(u_j - \tilde{u}_j)_j - (u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} + \int_{F_{ij}} \left\{ \frac{|(u_j - \tilde{u}_j)_j - (u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})}}{\text{dist}(s, \partial F_{ij})} \right\} ds .
\]

The first term of (4.40) is estimated as follows. Let \(Q_i\) be the \(L_2\) projection on \(X_i(F_{ij})\), the restriction of \(X_i(\partial \Omega_i)\) to \(F_{ij}\) with \(h_i\) -triangulation on \(F_{ij}\). Using the inverse inequality, the \(H^{1/2}\) and \(L_2\) stabilities of the \(L_2\) projection we have

\[
(4.41) \| (u_j - \tilde{u}_j)_j - (u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} \leq C \left\{ |Q_i((u_j)_j - (u_j)_j)|^2_{H^{1/2}(F_{ij})} + \\
+ |Q_i((u_j - \tilde{u}_j)_j)|^2_{H^{1/2}(F_{ij})} + |(u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} + |(u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} \right\} \leq \\
\leq C \left\{ \frac{1}{h_i} \| (u_j)_j - (u_j)_j \|^2_{L^2(F_{ij})} + \| (u_j - \tilde{u}_j)_j \|^2_{H^{1/2}(F_{ij})} + |(u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} + |(u_j - \tilde{u}_j)_j|^2_{H^{1/2}(F_{ij})} \right\} \leq \\
\leq C \left\{ \frac{1}{h_i} \| (u_j)_j - (u_j)_j \|^2_{L^2(F_{ij})} + \max((u_j)_j - (u_j)_j)^2 + (1 + \log \frac{H_j}{h_j})^2 \| \nabla (u_j)_j \|_{L^2(\Omega_j)} \right\}
\]

since \((\tilde{u}_j)_j\) and \((\tilde{u}_j)_j\) are linear on \(F_{ij}\) and \(F_{ij}\), respectively, for \(j \in \mathcal{E}^0_i\). The second term of right-hand side of last inequality of (4.41) is estimated as follows. Let \((\tilde{u}_j)_j\) be the average of \((u_j)_j\) on \(F_{ij}\). We obtain

\[
\max_{\partial F_{ij}}((u_j)_j - (u_j)_j)^2 \leq 3 \{ \max_{\partial F_{ij}}(Q_i((u_j)_j - (u_j)_j))^2 + \\
+ \max_{\partial F_{ij}}(Q_i((u_j)_j - (\tilde{u}_j)_j))^2 + \max_{\partial F_{ij}}(Q_i((u_j)_j - (u_j)_j))^2 \} \leq \\
\leq C \left\{ \frac{1}{h_i} \| (u_j)_j - (u_j)_j \|^2_{L^2(F_{ij})} + \max_{\partial F_{ij}}(Q_i((u_j)_j - (u_j)_j))^2 + \max_{\partial F_{ij}}(u_j - \tilde{u}_j)_j^2 \right\} .
\]
Hence, using a discrete Sobolev inequality, see [34], and the $H^{1/2}$ stability of the $L^2$ projection we obtain

$$\max_{\partial F_{ij}} |Q_i(u_j - \bar{u}_j)_{ij}|^2 \leq C(1 + \log \frac{H_i}{h_i})(|u_j|_{H^{1/2}(F_{ij})})^2 \leq C(1 + \log \frac{H_i}{h_i}) \|\nabla (u_j)_{ij}\|_{L^2(\Omega_j)}^2$$

and so

$$\max_{\partial F_{ij}} ((u_j)_i - (u_j)_j)^2 \leq C\left\{ \frac{1}{h_i} \| (u_j)_i - (u_j)_j \|_{L^2(\Omega_j)}^2 + + (1 + \log \frac{H_i}{h_i}) \|\nabla (u_j)_{ij}\|_{L^2(\Omega_j)}^2 \right\}. \quad (4.43)$$

Substituting this into (4.41), we get

$$\int_{F_{ij}} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{\text{dist}(s, \partial F_{ij})} ds \leq \leq C(1 + \log \frac{H_i}{h_i})^2\{ \|\nabla (u_j)_{ij}\|_{L^2(\Omega_j)}^2 + \frac{1}{h_i} \| (u_j)_i - (u_j)_j \|_{L^2(\Omega_j)}^2 \}. \quad (4.44)$$

We now estimate the second term of (4.40) as follows. In order to simplify notation we take $F_{ij}$ as the interval $[0, H]$. Note that

$$\int_0^{H/2} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds \leq \leq C\left\{ \int_0^{H/2} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds + \int_{H/2}^{H} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{(H - s)} ds \right\}. \quad (4.45)$$

Let us estimate the first term of right-hand side of (4.45). Let $h_i \leq h_j$ (the proof for $h_i > h_j$ is similar).

$$\int_0^{H/2} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds = \int_0^{h_i} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds + + \int_{h_i}^{H/2} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds \leq C\left\{ \int_0^{h_i} \frac{((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2}{s} ds + + \frac{1}{h_i} \| (u_j)_i - (u_j)_j \|_{L^2(\Omega_j)}^2 \right\} \leq \leq C(1 + \log \frac{H_i}{h_i}) \max_{\partial F_{ij}} ((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2 \leq \leq C(1 + \log \frac{H_i}{h_i}) \max_{\partial F_{ij}} ((u_j - \bar{u}_j)_i - (u_j - \bar{u}_j)_j)^2 \leq \leq C(1 + \log \frac{H_i}{h_i}) \{ \max_{\partial F_{ij}} ((u_j)_i - (u_j)_j)^2 + \max_{\partial F_{ij}} ((\bar{u}_j)_i - (\bar{u}_j)_j)^2 \} \leq \leq C(1 + \log \frac{H_i}{h_i}) \max_{\partial F_{ij}} ((u_j)_i - (u_j)_j)^2 \leq \leq C(1 + \log \frac{H_i}{h_i}) \{ \frac{1}{h_i} \| (u_j)_i - (u_j)_j \|_{L^2(\Omega_j)}^2 + \|\nabla (u_j)_{ij}\|_{L^2(\Omega_j)}^2 \}.$$

We have used (4.43) to obtain the last inequality. The second term of the right-hand side of (4.45) is also estimated by the right-hand side of the last inequality. Using
this estimate in (4.45) we get
\[
(4.46) \quad \int_{F_{ij}} \frac{(u_j - u_j^i)(u_j - u_j^i)}{\text{dist}(s, \partial F_{ij})} \, ds \leq C(1 + \log \frac{H}{R})^2 \{ \| \nabla (u_j)_j \|_{L^2(\Omega)}^2 + \frac{1}{h_j} \| (u_j)_i - (u_j)_j \|_{L^2(F_{ij})}^2 \}.
\]
Substituting (4.44) and (4.46) into (4.40) we get
\[
(4.47) \quad \| (u_j - u_j^i)(u_j - u_j^i) \|_{H^1_0(F_{ij})}^2 \leq C(1 + \log \frac{H}{R})^2 \{ \| \nabla (u_j)_j \|_{L^2(F_{ij})}^2 + \frac{1}{h_j} \| (u_j)_i - (u_j)_j \|_{L^2(F_{ij})}^2 \}.
\]
Substituting in turn this and (4.39) for \( i \) and \( j \) into (4.38) we get
\[
(4.48) \quad \rho_i \| (v_i)_i \|_{H^1_0(F_{ij})}^2 \leq C(1 + \log \frac{H}{R})^2 \{ \hat{d}_i(u_i, u_i) + \hat{d}_j(u_j, u_j) \}
\]

It remains to estimate the second term of the right-hand side of (4.36). The case \( F_{ij} \) where \( j \in E_i^0 \) is trivial. For the case \( F_{ij} \) such that \( j \in E_i^1 \) we have, see (4.22) - (4.23),
\[
(4.49) \quad \rho_i \| (v_i)_i \|_{L^2(F_{ij})}^2 = \frac{\rho_i \rho_j^{2\gamma}}{(\rho_i^2 + \rho_j^2)^2} \times \\
\quad \times \| [u_i - u_i^j_i] \|_{L^2(F_{ij})}^2 + \| [u_j - u_j^i_j] - (u_i - u_i^j_i) \|_{L^2(F_{ij})}^2 \leq 2 \rho_i \| (u_i)_i - (u_i)_j \|_{L^2(F_{ij})}^2 + 2 \rho_j \| (u_j)_i - (u_j)_j \|_{L^2(F_{ij})}^2
\]

since \((u_i)_i = (u_j)_i\) and \((u_j)_j = (u_i)_j\) on \( F_{ij} \) and \( F_{ji} \), respectively. Hence
\[
(4.50) \quad \sum_{j \in E_i^1} \frac{\rho_i \delta}{h_j} \| (v_i)_i - (v_i)_j \|_{L^2(F_{ij})}^2 \leq C \sum_{j \in E_i^1} \frac{\delta}{h_j} \{ \rho_i \| (u_i)_i - (u_i)_j \|_{L^2(F_{ij})}^2 + \rho_j \| (u_j)_i - (u_j)_j \|_{L^2(F_{ij})}^2 \}
\]

Substituting (4.48) and (4.50) into (4.36) we get
\[
(4.51) \quad \hat{d}_i(v_i, v_i) \leq C(1 + \log \frac{H}{R})^2 \{ \hat{d}_i(u_i, u_i) + \sum_{j \in E_i} \hat{d}_j(u_j, u_j) \}.
\]

Summing the inequalities (4.51) for \( i \) from 1 to \( N \) and noting that the number of edges of each subdomain can be bounded independently of \( N \), we obtain (4.35) and (4.34).

The proof also works with minor modifications for the case when \( F_{ij} \) is a continuous curve of intervals. For that, we should consider discrete Sobolev tools for non straight edges, see for instance [23], and interpret \( I_{F_{ij}}(u_i)_i \) and \( I_{F_{ij}}(u_i)_j \) as the linear function with respect to parametrized path on the edge defined by the nodal value of \((u_i)_i \) or \((u_i)_j \) at \( x \in \partial F_{ij} \) and \( \partial F_{ji} \).
5. Implementation. The problem (4.15) can be solved efficiently by the pre-conditioned conjugate gradient method with the pre-conditioner $M$ defined in (4.27). To simplify the presentation we only discuss the Richardson’s method. For the system of algebraic equations, see (4.15),

$$F\lambda^* = g$$

(5.1)

the Richardson iterative method is of the form: with given $\lambda_0$ and for $k = 0, 1, \cdots$

$$\lambda_{k+1} = \lambda_k - \tau_{opt} M^{-1} (F\lambda_k - g)$$

(5.2)  

where $\tau_{opt} = 2/(C(1 + \log H/h)^2 + 1))$, see (4.28). We need to compute first

$$\tilde{r}_k := F\lambda_k - g = B_\Delta \tilde{S}^{-1}(B_\Delta^T\lambda_k - \tilde{g}_\Delta) = B_\Delta \tilde{S}^{-1}\tilde{g}_k$$

(5.3)

and then, see (4.27),

$$r_k := M^{-1}\tilde{r}_k = \sum_{i=1}^{N} B_\Delta^{(i)} D_\Delta^{(i)} S'_{t,\Delta} D_\Delta^{(i)} (B_\Delta^{(i)})^T \tilde{r}_k$$

(5.4)

where $\tilde{r}_k = \{\tilde{r}_{i,k}\}_{i=1}^{N} \in W_\Delta(\Gamma')$ with $\tilde{r}_{i,k} \in W_{i,\Delta}(\Gamma_i')$.

To compute $\tilde{r}_k$ we need to solve a system

$$\tilde{S}\tilde{v}_k = \tilde{g}_k.$$  

(5.5)

Note that $\tilde{S}$ is a global matrix but with very weak couplings only through the corners of substructures $\Omega_i$ for $1 \leq i \leq N$. Hence, the system with $\tilde{S}$ is solved by a special algorithm based on the Cholesky factorization. For the conforming case this algorithm is described in the book [34], see pp. 166-167. We modify this algorithm to solve (5.5). The main modification corresponds to the fact that in our case in a common corner of substructures $\Omega_i$ we have multiple unknowns while in the conforming case we have only one value. A computation of $B_\Delta v$, see (4.9) and (4.10), for a given $v = \{v_i\}_{i=1}^{N} \in W_\Delta(\Gamma')$, reduces to multiply the rectangular matrices $B_\Delta^{(i)}$ with entries $\{0, 1, -1\}$ by the subvector of $v_i$ belonging to $W_{i,\Delta}(\Gamma_i')$.

To compute $M^{-1}\tilde{r}_k$, see (5.4), we need to compute for $1 \leq i \leq N$

$$S'_{t,\Delta} D_\Delta^{(i)} (B_\Delta^{(i)})^T \tilde{r}_k =: S'_{t,\Delta} \tilde{v}_{i,k}.$$  

(5.6)

A computation of $\tilde{v}_{i,k} := D_\Delta^{(i)} (B_\Delta^{(i)})^T \tilde{r}_k$ reduces to a multiplication of $\tilde{r}_k$ by $(B_\Delta^{(i)})^T$ and then by the diagonal scaling matrix $D_\Delta^{(i)}$, see (4.19). In turn, a computation of $S'_{t,\Delta} \tilde{v}_{i,k}$ is reduced to solving a local problem defined on $\Gamma_i'$, see (4.18), with zero values of $\tilde{v}_i$ at the corners $V_i'$ in $\Omega_i'$. These problems involve the solution of the problem $S'_{i}$ on each $\Omega_i'$ with Dirichlet data on $\Gamma_i'$. We point out that the local problems are independent so they can be solved in parallel.

Finally compute

$$\lambda_{k+1} = \lambda_k - \tau_{opt} r_k$$

with the computed above $r_k$. 


6. Numerical experiments. In this section, we present numerical results for solving the linear system (4.15) with the left preconditioner (4.27). We show that the lower and upper bounds of Theorem 4.3 are reflected in the numerical tests. In particular we show that the constant $C$ in (4.28) does not depend on $h_i$, $h_i/h_j$, $H_i$, and the jumps of $\rho_i$.

![Fig. 6.1. A 4x4 subdomain partition with 4x4 structured local mesh for the black subdomains and 3x3 structured local mesh for the red subdomains.](image)

We consider the domain $\Omega = (0,1)^2$ and divide into $N = M \times M$ squares subdomains $\Omega_i$ of size $H = 1/M$. Inside each subdomain $\Omega_i$ we generate a structured triangulation with $n_i$ subintervals in each coordinate direction and apply the discretization presented in Section 2.2 with penalty term $\delta = 4$. In the numerical experiments we use a red and black checkerboard type of subdomain partition, where the most bottom-left subdomain has a black color. On the black subdomains we let $n_i = n_b = 2 \times 2^{L_b}$ and on the red subdomains we let $n_i = n_r = 3 \times 2^{L_r}$, where $L_b$ and $L_r$ are integers denoting the number of refinements inside each subdomain $\Omega_i$. See Figure 6.1 for the case $M = 4$, $L_b = 1$ and $L_r = 0$. Hence the local mesh sizes are $h_b = \frac{1}{M n_b}$ and $h_r = \frac{1}{M n_r}$, respectively. We solve the second order elliptic problem $-\text{div}(\rho(x)\nabla u^*_x) = 1$ in $\Omega$ with homogeneous Dirichlet boundary conditions $u^* = 0$. In the numerical experiments, we run PCG until the $l_2$ initial residual is reduced by a factor of $10^8$. In all the experiments we consider $\beta = 1$, see (4.19).

In the first test of experiments we consider the constant coefficient case $\rho = 1$. We consider different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$, therefore keeping constant the mesh ratio $h_b/h_r = 3/2$. Table 6.1 lists the number of PCG iterations and in parenthesis the condition number estimate of the preconditioned system. As expected from the analysis, the condition numbers appear to be independent of the number of subdomains and grow by a two-logarithmically factor when the size of the local problems increases. As expected from the theory, the lower bounds estimates are always very close to one, therefore, we do not show in the tables.

We now consider the discontinuous coefficients case where we set $\rho_b = 1$ on the black substructures and we vary $\rho_r$ on the red substructures. The substructures partition is kept fixed to $4 \times 4$. Table 6.2 lists the results on runs for different values of $\rho_r$ and for different levels of refinements $L_r$ on the red substructures. On the black substructures $n_b = 2$ is kept fixed. The performance of the preconditioner is robust with respect to the coefficients and the mesh ratio $h_b/h_r$ as predicted.
Table 6.1
Number of iterations, condition numbers (in parenthesis) for different sizes of coarse and local problems and with constant coefficient $\rho_b = \rho_r = 1$ and $L_b = L_r$.

<table>
<thead>
<tr>
<th>$L_r \downarrow M \rightarrow$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10 (2.22)</td>
<td>12 (2.94)</td>
<td>14 (3.36)</td>
<td>15 (3.50)</td>
</tr>
<tr>
<td>1</td>
<td>12 (2.94)</td>
<td>15 (3.88)</td>
<td>16 (4.15)</td>
<td>17 (4.26)</td>
</tr>
<tr>
<td>2</td>
<td>14 (3.27)</td>
<td>15 (4.51)</td>
<td>17 (4.86)</td>
<td>18 (4.98)</td>
</tr>
<tr>
<td>3</td>
<td>15 (3.34)</td>
<td>15 (5.34)</td>
<td>19 (5.81)</td>
<td>19 (5.94)</td>
</tr>
<tr>
<td>4</td>
<td>15 (3.34)</td>
<td>17 (6.39)</td>
<td>21 (6.99)</td>
<td>21 (7.14)</td>
</tr>
</tbody>
</table>

Table 6.2
Number of iterations and condition numbers (in parenthesis) for different values of the coefficient $\rho_r$ and local meshes with $L_r$ refinements on the red substructures. On black substructures the coefficient $\rho_b = 1$ and $L_b = 0$ are kept fixed. The substructure partition is also kept fixed to $4 \times 4$.

<table>
<thead>
<tr>
<th>$L_r \downarrow \rho_r \rightarrow$</th>
<th>1000</th>
<th>10</th>
<th>1</th>
<th>0.1</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6 (1.18)</td>
<td>9 (1.79)</td>
<td>12 (2.94)</td>
<td>9 (1.91)</td>
<td>5 (1.18)</td>
</tr>
<tr>
<td>1</td>
<td>6 (1.15)</td>
<td>10 (1.79)</td>
<td>14 (3.56)</td>
<td>11 (2.30)</td>
<td>5 (1.28)</td>
</tr>
<tr>
<td>2</td>
<td>5 (1.13)</td>
<td>10 (2.05)</td>
<td>17 (4.39)</td>
<td>11 (2.47)</td>
<td>5 (1.39)</td>
</tr>
<tr>
<td>3</td>
<td>5 (1.11)</td>
<td>11 (2.19)</td>
<td>17 (4.54)</td>
<td>12 (2.44)</td>
<td>5 (1.50)</td>
</tr>
<tr>
<td>4</td>
<td>5 (1.11)</td>
<td>11 (2.22)</td>
<td>18 (4.72)</td>
<td>11 (2.33)</td>
<td>5 (1.62)</td>
</tr>
</tbody>
</table>

7. Conclusions. In this paper we consider a composite FE/DG discretization for handling second order elliptic equations with discontinuous coefficients and non-matching meshes across subdomains. We design and analyze a FETI-DP preconditioner for the associated discrete system and prove (see Theorem 4.3 and Lemmas 4.4 and 4.5) that the condition number of this preconditioned system only grows two-logarithmically with respect to $H_i/h_i$, and does not depend on local mesh size $h_i$, a size of the subdomains $H_i$, and jumps of coefficient $\rho_i$ and the mesh ratio $h_i/h_j$ across subdomain edges $F_{ij}$. The numerical tests confirm the theoretical results. Up to how the diagonal weighting matrices (4.19) are chosen and the availability of Neumann matrices of the subdomains, the method is fully algebraic. The algorithm can also be extended to 3-D cases by selecting primals which impose continuity at the corners of subdomains and on averages across the edges of subdomains, and its mathematical analysis needs a lot of details, therefore, it is not discussed in the paper. It will be considered separately. The algorithm also can be extended to full DG discretization, and it will be discussed elsewhere.

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REFERENCES


