Group Isomorphisms
MME 529 Worksheet for May 23, 2017
William J. Martin, WPI

Goal: Illustrate the power of abstraction by seeing how groups arising in different contexts are really the same.

There are many different kinds of groups, arising in a dizzying variety of contexts. Even on this worksheet, there are too many groups for any one of us to absorb. But, with different teams exploring different examples, we should – as a class – discover some justification for the study of groups in the abstract.

The Integers Modulo n: With John Goulet, you explored the additive structure of \( \mathbb{Z}_n \). Write down the addition table for \( \mathbb{Z}_5 \) and \( \mathbb{Z}_6 \). These groups are called cyclic groups: they are generated by a single element, the element 1, in this case. That means that every element can be found by adding 1 to itself an appropriate number of times.

The Group of Units Modulo n: Now when we look at \( \mathbb{Z}_n \) using multiplication as our operation, we no longer have a group. (Why not?) The group \( U(n) = \{ a \in \mathbb{Z}_n \mid \gcd(a, n) = 1 \} \) is sometimes written \( \mathbb{Z}_n^* \) and is called the group of units modulo \( n \). An element in a number system (or ring) is a “unit” if it has a multiplicative inverse. Write down the multiplication tables for \( U(6) \), \( U(7) \), \( U(8) \) and \( U(12) \).

The Group of Rotations of a Regular n-Gon: Imagine a regular polygon with \( n \) sides centered at the origin \( O \). Let \( e \) denote the identity transformation, which leaves the polygon entirely fixed and let \( a \) denote a rotation about \( O \) in the counterclockwise direction by exactly \( 360/n \) degrees (\( 2\pi/n \) radians). Then, applying this twice, we find that \( a^2 \) is a counterclockwise rotation by \( 720/n \) degrees. Find the order of \( a \) — the smallest positive integer \( k \) such that \( a^k = e \) — and write down the Cayley table for the group \( \mathbb{Z}_n = \{ e, a, a^2, \ldots, a^{k-1} \} \).

The Dihedral Group \( D_n \): A regular polygon with \( n \) sides has \( 2n \) symmetries, including \( n \) rotations and \( n \) reflections. Write down the Cayley tables for \( D_4 \) and \( D_5 \).

The Group \( GL(2, \mathbb{F}) \) of Invertible 2 \times 2 Matrices: For \( \mathbb{F} = \mathbb{Z}_2 \), work out the Cayley table for the group of all \( 2 \times 2 \) matrices \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with entries from \( \mathbb{F} \) satisfying \( ad - bc \neq 0 \). The next case is \( \mathbb{F} = \mathbb{Z}_3 \), but that Cayley table is too large to work out by hand since it has \( 8 \cdot 6 = 48 \) elements. Discuss the case where \( \mathbb{F} \) is the field of real numbers or rational numbers.

The External Direct Product: If \( G \) and \( H \) are groups, then the Cartesian product \( G \times H \) becomes a group by applying the two operations coordinatewise. Gallian denotes this group by \( G \oplus H \). For example, \( \mathbb{Z}_2 \oplus \mathbb{Z}_4 \) is isomorphic to \( \mathbb{Z}_6 \): \( (1, 1) + (1, 1) = (0, 2) \),
\[(1, 1) + (1, 1) + (1, 1) = (1, 0), \] etc. Work out the addition tables for \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) and \(\mathbb{Z}_2 \oplus \mathbb{Z}_4\). Diagrammatically describe the Cayley table for \(\mathbb{Z}_2 \oplus S_3\).

**The Symmetry Group of a Graph:** The graph \(P\) below is called the *Petersen graph*. It has ten vertices and 15 edges; two vertices joined by an edge are said to be adjacent. An automorphism of \(P\) is a permutation of the vertices of \(P\) which preserves adjacency. That is,

\[\varphi : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 10\}\]

bijectively (i.e., the function is both one-to-one and onto) in such a way that vertex \(i\) is adjacent to vertex \(j\) if and only if \(\varphi(i)\) is adjacent to \(\varphi(j)\). How many symmetries does the Petersen graph have?

\[P\]

**The Quaternion Group:** The field of complex numbers \(\mathbb{C}\) consists of all expressions of the form \(a + bi\) where \(a\) and \(b\) are real numbers and \(i\) is a symbol which satisfies only the rule \(i^2 = -1\). The nonzero complex numbers form an abelian group under multiplication and the complex numbers of modulus one — i.e., those \(z = a + bi\) with \(|z|^2 = a^2 + b^2 = 1\) — form what is called the “circle group”. Consider the subgroup of the circle group consisting of \(\pm 1\) and \(\pm i\). Construct a Cayley table (multiplication table, in this case) for this group of order four. What familiar group is it isomorphic to?

Now consider the division ring of *quaternions*

\[\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}\]

where now we imagine size distinct square roots of \(-1\). We define symbols \(i, j\) and \(k\) satisfying

\[i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j.\]

Work out the multiplication table for the “quaternion group”

\[Q_8 = \{1, -1, i, -i, j, -j, k, -k\}\]

**The Sporadic Finite Simple Groups:** In addition to cyclic groups of prime order, there are seventeen infinite families of finite simple groups\(^1\) and exactly 26 exceptions. These 26

1A group is simple if it does not “break up” further into smaller groups – more precisely, \(G\) is simple if every nontrivial group homomorphism \(G \to H\) is one-to-one.
“sporadic finite simple groups” were discovered between 1861 and 1976 or so. Use MAPLE’s "GroupTheory" package to find the order (size) of as many of these as you can. A full list can be found in Wikipedia.