LP Assignment 5

DUE DATE: Tuesday, September 25, 2018, at the beginning of class.

Please carefully review the assignment presentation rules on the back of this sheet.

Provide neat and careful solutions to the following five problems:

1.) Your company uses three resources — snot, spit and filth — to manufacture three products — goodness, compassion and cheer. Let

- $x_1$ denote the number of expressions of goodness to be produced;
- $x_2$ denote the number of acts of compassion to be produced;
- $x_3$ denote the number of bundles of cheer to be produced.

With 120 and 240 gallons, respectively, of snot and spit and and 180 tons of filth available, the following LP formulation uses per-unit profit for each product in order to maximize overall profit (in euros) subject to resource constraints:

\[
\begin{align*}
\text{maximize} \quad & \quad 20x_1 + 30x_2 + 40x_3 \\
\text{subject to} \quad & \quad x_1 + x_2 + x_3 \leq 120 \\
& \quad 2x_1 - x_2 + 3x_3 \leq 240 \\
& \quad 2x_1 + 4x_2 \leq 180 \\
& \quad x_1, \; x_2, \; x_3 \geq 0
\end{align*}
\]

The optimal dictionary is

\[
\begin{array}{cccc}
\zeta & = & 4500 & 35x_1 & 65x_4 & 5x_5 \\
x_2 & = & 30 & -\frac{1}{3}x_1 & -\frac{3}{4}x_4 & -\frac{3}{4}x_5 \\
x_3 & = & 90 & -\frac{3}{4}x_1 & -\frac{1}{4}x_4 & -\frac{3}{4}x_5 \\
x_6 & = & 60 & -x_1 & +3x_4 & -x_5
\end{array}
\]

(a) Write down the shadow prices (optimal dual variable values) for each of the three resources. For each, write a sentence explaining what this means in terms of the manufacturing problem.

(b) Suppose that an extra gallon of snot becomes available to you. How much would you be willing to pay for it and why? Up to what limit is this price sensible? Do the same for spit and filth. (Do not pivot.)
(c) Suppose that the “per-expression” profit for goodness changes from $20 to $35. Would you start producing goodness or not? Explain. (Do not pivot.)

(d) Next, suppose that, after optimizing, you learn that the profit per unit for cheer has diminished from 40 to 10. Write down the modified final dictionary. If it is no longer optimal, then pivot to optimality.

(e) Suppose that, after solving the original LP, the amount of spit available is reduced from 240 gallons to 160 gallons. Work out the modified final dictionary. Apply the dual simplex method to pivot from this dictionary to optimality.

(f) What is the smallest number of gallons $b_2$ of spit for which the basis $\{2, 3, 6\}$ is optimal?

2.) Consider the LP

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + 2x_2 + 3x_3 \\
\text{subject to} & \\
& x_1 + x_2 + x_3 \leq 14 \\
& 4x_1 + x_2 + 2x_3 \leq 40 \\
& 2x_1 + x_2 + x_3 \leq 21 \\
& x_1, x_2, x_3 \geq 0 \\
\end{align*}
\]

with optimal dictionary

\[
\begin{array}{ccccc}
\zeta & = & 48 & -\frac{1}{2}x_2 & -2x_4 & -\frac{1}{2}x_5 \\
x_3 & = & 8 & -\frac{3}{2}x_2 & -2x_4 & +\frac{1}{2}x_5 \\
x_6 & = & 1 & -\frac{1}{2}x_2 & +\frac{1}{2}x_5 \\
x_1 & = & 6 & +\frac{1}{2}x_2 & +x_4 & -\frac{1}{2}x_5 \\
\end{array}
\]

(a) For each variable (decision variables as well as slack variables), find the range of optimality for the current basis relative to objective coefficient $c_j$. That is, for what real numbers $c_j$ is the current dictionary still optimal? (Be sure to give your answer in terms of the original data, $c_j$, not relative to the value $\bar{c}_j$ appearing in this dictionary.)

(b) For decision variables $x_1$, $x_2$ and $x_3$, tell me what must be done next (a simplex pivot or a dual simplex pivot?) when optimality is comprised. Do not pivot. Simply give the entering and leaving variable.

(c) Comparing the original problem and the optimal dictionary above, for each entry of vector $b = (14, 40, 21)$, find the range of optimality for the current basis relative to right-hand side value $b_i$. For what real numbers $b_i$ is the (slightly modified) current dictionary still optimal? When optimality is compromised, tell me what must be done next (a simplex pivot or a dual simplex pivot?). Do not pivot, but be sure to give the entering and leaving variable for the first pivot on your way back to optimality.
3.) Apply the Dual Simplex Method on the following problem.

$$\text{max} \ -12x_1 - 8x_2 - 12x_3$$

s.t. 
$$-2x_1 - x_2 - x_3 \leq -1$$
$$-x_1 - x_2 - 2x_3 \leq -3$$
$$x_1, x_2, x_3 \geq 0$$

4.) Consider the LP problem

$$\text{maximize} \ -4x_1 - 4x_2 - 15x_3$$

subject to 
$$x_1 - x_2 - 6x_3 \leq -2$$
$$-3x_1 + 2x_2 + 14x_3 \leq -6$$
$$x_1, x_2, x_3 \geq 0$$

(a) Optimize this problem using the dual simplex method.

(b) Write down the dual of this LP.

(c) Optimize this dual problem using the ordinary simplex method. (NOTE: Recall that our simplex algorithm only maximizes, so use the fact that $\min \omega$ has the same effect as $\max -\omega$ except that the answer has its sign reversed.)

5.) Suppose $B$ is the optimal basis for an LP problem with $n$ variables and $m$ constraints in standard form $\max c^\top x$ subject to $Ax \leq b$, $x \geq 0$. Consider now, instead, the region of all vectors $b'$ for which $B$ is still the optimal basis for the modified problem

$$\max c^\top x \text{ subject to } Ax \leq b', x \geq 0.$$ 

Now, for this inverted problem — where $A$, $c$ and $B$ are fixed, $b' \in \mathbb{R}^m$ is varying, and $x$ given by the dictionary corresponding to basis $B$ — determine whether or not the objective function is bounded. Can you give an example where $c^\top x$ is bounded? Can you give an example where $c^\top x$ is unbounded? Why or why not? Explain.

BASIC RULES FOR LP ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only one side of each sheet of paper;

III) Show your work. Explain your answers using FULL SENTENCES;

IV) Late assignments will not, in general, be accepted for credit.