MA3231 Linear Programming
W. J. Martin
September 27, 2014

Sample Solutions
LP Assignment 3

1. (Exercise 2.19) Suppose \( p_1/q_1 < p_2/q_2 < \cdots < p_n/q_n \) where the \( p_i \) and the \( q_i \) are sequences of positive real numbers each summing to one. For \( \beta > 0 \), the optimal solution to the linear programming problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} p_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} q_j x_j \leq \beta \\
& \quad 0 \leq x_j \leq 1 \quad (1 \leq j \leq n)
\end{align*}
\]

is as follows. Locate the smallest \( k \) \( (0 \leq k \leq n) \) such that

\[
q_{k+1} + q_{k+2} + \cdots + q_{n-1} + q_n \leq \beta.
\]

Then choose

\[
x_j = \begin{cases} 
1, & j > k \\
0, & j < k \\
(\beta - q_{k+1} - \cdots - q_n)/q_k, & j = k
\end{cases}
\]

(1)

I claim that this is an optimal solution to the problem. If we introduce slack variables \( u \) for the first constraint and \( w_1, \ldots, w_n \) for the constraints \( x_j \leq 1 \) \( (1 \leq j \leq n) \), we have initial dictionary

\[
\begin{array}{cccccc}
\zeta & = & +p_1 x_1 & +p_2 x_2 & \cdots & +p_n x_n \\
u & = & \beta & -q_1 x_1 & -q_2 x_2 & \cdots & -q_n x_n \\
w_1 & = & 1 & -x_1 \\
w_2 & = & 1 & -x_2 \\
\vdots & & \vdots & \ddots & \vdots \\
w_n & = & 1 & \cdots & -x_n
\end{array}
\]

Our claim is that the optimal basis consists of variables

\[\{x_k, x_{k+1}, \ldots, x_n, w_1, \ldots, w_k\}\]
For basis 
\[ \{ u, x_{k+1}, \ldots, x_n, w_1, \ldots, w_k \}, \]
the corresponding dictionary is easily derived by swapping \( x_j \) with \( w_j \) in each constraint \( w_j = 1 - x_j \) \( (k < j \leq n) \):

\[
\begin{align*}
\zeta &= \gamma + p_1 x_1 + \cdots + p_k x_k - p_{k+1} w_{k+1} - \cdots - p_n w_n \\
u &= \delta - q_1 x_1 - \cdots - q_{k-1} x_{k-1} + q_{k+1} w_{k+1} + \cdots + q_n w_n \\
w_1 &= 1 - x_1 \\
\vdots & \vdots \\
w_k &= 1 - x_k \\
x_{k+1} &= 1 - w_{k+1} \\
\vdots & \vdots \\
x_n &= 1 - w_n
\end{align*}
\]

where we have abbreviated
\[
\gamma = p_{k+1} + \cdots + p_n, \quad \delta = \beta - q_{k+1} - \cdots - q_n.
\]

Now we pivot \( x_k \) in and, by our choice of \( k \), the ratio computation gives \( u \) as the leaving variable instead of \( w_k \). The final dictionary is

\[
\begin{align*}
\zeta' &= \gamma' + \left( \frac{p_1 q_k - p_k q_1}{q_k} \right) x_1 + \cdots + \left( \frac{p_{k-1} q_k - p_k q_{k-1}}{q_k} \right) x_{k-1} - \frac{1}{q_k} u + \left( \frac{p_k q_{k+1} - p_{k+1} q_k}{q_k} \right) w_{k+1} + \cdots + \left( \frac{p_n q_n - p_n q_k}{q_k} \right) w_n \\
x_k &= \delta' - q_1 x_1 - \cdots - q_{k-1} x_{k-1} - u + q_{k+1} w_{k+1} + \cdots + q_n w_n \\
w_1 &= 1 - x_1 \\
\vdots & \vdots \\
w_k &= 1 - x_k \\
x_{k+1} &= 1 - w_{k+1} \\
\vdots & \vdots \\
x_n &= 1 - w_n
\end{align*}
\]

where we have omitted the messy \( x_k \) equation

\[
x_k = \frac{\delta - q_1 x_1 - \cdots - q_{k-1} x_{k-1} - u + q_{k+1} w_{k+1} + \cdots + q_n w_n}{q_k}
\]

and the positive scalars \( \delta' = \delta / q_k \) and \( \gamma' = \gamma + \frac{p_k \delta}{q_k} \).

To verify that this dictionary is optimal, we must show that all objective coefficients are non-positive. But the hypothesis

\[
p_1 / q_1 < \cdots < p_{k-1} / q_{k-1} < p_k / q_k < p_{k+1} / q_{k+1} < \cdots < p_n / q_n
\]

gives us \( p_i - p_k q_i < 0 \) for \( i < k \) and \( p_k q_j - p_j < 0 \) for \( j > k \), which is exactly what we need. \( \square \)
2. (Exercise 3.4) Given an LP of the form

\[
\begin{align*}
\text{max} & \quad c^\top x \\
\text{s.t.} & \quad Ax \leq 0, \quad x \geq 0,
\end{align*}
\]

either the zero vector \(x = 0\) is an optimal solution or the problem is unbounded.

**Proof:** Observe that \(x = 0\) is a feasible solution to the problem. (This is easy to check by substituting \(x = 0\) into the constraints.) We consider two cases, either \(x = 0\) is an optimal solution, or it is not. If it is optimal, then we have one of the outcomes we seek and we are done.

So suppose now that \(x = 0\) is not optimal. That means we can do better. So there is some feasible solution \(y\) with better objective value. Since \(c^\top 0 = 0\), we then have

\[
Ay \leq 0, \quad y \geq 0, \quad c^\top y > 0.
\]

Now, for any positive scalar \(\alpha > 0\), consider the vector \(\alpha y\). We easily check that this is a feasible solution:

\[
A(\alpha y) = \alpha (Ay) \leq \alpha 0
\]

since \(\alpha > 0\), giving \(A(\alpha y) \leq 0\), and likewise

\[
\alpha y \geq \alpha 0 = 0.
\]

But this new solution has objective value

\[
c^\top (\alpha y) = \alpha c^\top y
\]

which can be made as large as we like by choosing \(\alpha\) sufficiently large since \(c^\top y > 0\). So the problem is unbounded. \(\square\)

3. (a) The Klee-Minty problem for \(n = 2\) is

\[
\begin{align*}
\text{maximize} & \quad 10x_1 + x_2 \\
\text{subject to} & \quad x_1 \leq 1 \\
& \quad 20x_1 + x_2 \leq 100 \\
& \quad x_1, \quad x_2 \geq 0
\end{align*}
\]

(b) The feasible region is a slightly irregular quadrilateral (plus interior) in the plane, but I won’t draw it here.

(c) If we insert slack variables \(w_1\) and \(w_2\) and apply the simplex method, we find the following sequence of dictionaries:

\[
\begin{align*}
\zeta &= 10x_1 + x_2 \\
w_1 &= 1 - x_1 \\
w_2 &= 100 - 20x_1 - x_2
\end{align*}
\]

\[3\]
\begin{align*}
\zeta &= 10 \quad -10w_1 \quad +x_2 \\
x_1 &= 1 \quad - \quad w_1 \\
w_2 &= 80 \quad +20x_1 \quad -x_2 \\
\zeta &= 90 \quad +10w_1 \quad -w_2 \\
x_1 &= 1 \quad - \quad w_1 \\
x_2 &= 80 \quad +20x_1 \quad -w_2 \\
\zeta &= 100 \quad -10x_1 \quad -w_2 \\
w_1 &= 1 \quad - \quad x_1 \\
x_2 &= 80 \quad -20x_1 \quad -w_2 \\

\text{In this way, we visit every corner of the feasible region before reaching the optimal solution.}
\end{align*}

4. Write a mathematical argument that determines all optimal solutions to

\begin{align*}
\text{maximize} & \quad 4x_1 + 5x_2 + 8x_3 + 9x_4 \\
\text{subject to} & \quad 4x_1 + 5x_2 \leq 100 \\
& \quad 8x_3 + 9x_4 \leq 100 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}

\textbf{Solution:} We observe that the sum of the two constraints is

\[ 4x_1 + 5x_2 + 8x_3 + 9x_4 \leq 200. \]

Since every feasible solution satisfies both constraints, every feasible solution satisfies this inequality. Thus the objective function is bounded above by 200. The only way to achieve objective value 200 is to force

\begin{align*}
4x_1 + 5x_2 &= 100 \\
8x_3 + 9x_4 &= 100.
\end{align*}

So the set of optimal solutions is a rectangle in 4-space, consisting of all points \((x_1, x_2, x_3, x_4)\) where \((x_1, x_2)\) is on the line segment

\[ \{(x_1, x_2) \mid x_1 \geq 0, \ x_2 \geq 0, \ 4x_1 + 5x_2 = 100 \} \]

and where \((x_3, x_4)\) is on the line segment

\[ \{(x_3, x_4) \mid x_3 \geq 0, \ x_4 \geq 0, \ 8x_3 + 9x_4 = 100 \}. \]
5. Find the dual of the following LP

\[
\text{maximize} \quad 9x_1 - 5x_2 \\
\text{subject to} \quad 6x_1 + 2x_2 \leq 10 \\
\quad \quad \quad \quad \quad -7x_1 - 3x_2 \leq 20 \\
\quad \quad \quad \quad \quad 8x_1 - 4x_2 \leq 30 \\
\quad \quad \quad \quad \quad x_1, \quad x_2 \geq 0
\]

The dual problem is

\[
\text{minimize} \quad 10y_1 + 20y_2 + 30y_3 \\
\text{subject to} \quad 6y_1 - 7y_2 + 8y_3 \geq 9 \\
\quad \quad \quad \quad \quad 2y_1 - 3y_2 - 4y_3 \geq -5 \\
\quad \quad \quad \quad \quad y_1, \quad y_2, \quad y_3 \geq 0
\]

Some notes:

- Since our original is a maximization problem, our dual goal is to find upper bounds on the objective function. Since we want the best upper bound, we want to minimize \(w\) subject to the restriction that

\[
t_1x_1 + t_2x_2 \leq w
\]

is a valid combination of the original constraints.

- We multiply the first primal constraint by \(y_1\), the second by \(y_2\) and the third by \(y_3\) to get

\[
y_1(6x_1 + 2x_2) + y_2(-7x_1 - 3x_2) + \\
y_3(8x_1 - 4x_2) = t_1x_1 + t_2x_2.
\]

Thus we have

\[
t_1 = 6y_1 - 7y_2 + 8y_3 \\
t_2 = 2y_1 - 3y_2 - 4y_3.
\]

- Since we want upper bounds on the objective function, we need \(t_1 \geq 9\) and \(t_2 \geq -5\) since \(x_1 \geq 0\) and \(x_2 \geq 0\), respectively. This gives us our constraints.

- Since we want our combination of constraints to take the form

\[
t_1x_1 + t_2x_2 \leq w
\]

we cannot flip any of our primal inequality constraints, so none of \(y_1, y_2, y_3\) can be negative.
• This gives us an upper bound $w$, which is a combination of the right-hand side values of the constraints:

$$w = 10y_1 + 20y_2 + 30y_3.$$