Combinatorics, D Term, 2008
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Combinatorics Practice Problems

PRACTICE ONLY: These are good problems to practice for Test 2. Do not hand in.

N.B. Test 2 will be held on Thursday, April 10, in class.

[1] Please match the generating function to the recurrence. To develop good skills in both directions, be sure to solve a few of them each way: some starting from the left and some starting from the right.

(a) $A(x) = \frac{x+1}{1-x-x^2}$

(b) $B(x) = \frac{x+1}{1-2x-x^2}$

(c) $C(x) = \frac{2x}{1-x-x^2}$

(d) $D(x) = \frac{2x}{1-2x-x^2}$

(e) $E(x) = \frac{x^3+x+1}{1-x^3}$

(f) $F(x) = \frac{x^2-x-2}{1-x^3}$

(g) $G(x) = \frac{2x+3}{(1-x)(1-x^2)}$

(h) $H(x) = \frac{3-x}{(1-x)(1-x^2)}$

(i) $I(x) = \frac{x^2+1}{(1-x)(1-x^2)}$

(j) $J(x) = \frac{x^2-3x}{(1-x)(1-x^2)}$

(k) $K(x) = \frac{1+2x+3x^2}{(1-x-x^3)}$

(l) $L(x) = \frac{1-2x^2}{(1-x-x^3)}$

1. $a_0 = 1, a_1 = 3, a_2 = 6,$ $\quad a_n = a_{n-1} + a_{n-3}$ $(n \geq 3)$
2. $a_0 = 0, a_1 = 2,$ $\quad a_n = 2a_{n-1} + a_{n-2}$ $(n \geq 2)$
3. $a_0 = 1, a_1 = 1, a_2 = 3,$ $\quad a_n = a_{n-1} + a_{n-2} - a_{n-3}$ $(n \geq 3)$
4. $a_0 = 1, a_1 = 3,$ $\quad a_n = 2a_{n-1} + a_{n-2}$ $(n \geq 2)$
5. $a_0 = -2, a_1 = -1, a_2 = 1,$ $\quad a_n = a_{n-3}$ $(n \geq 3)$
6. $a_0 = 0, a_1 = 2,$ $\quad a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$
7. $a_0 = 1, a_1 = 1, a_2 = -1,$ $\quad a_n = a_{n-1} + a_{n-3}$ $(n \geq 3)$
8. $a_0 = 1, a_1 = 2,$ $\quad a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$
9. $a_0 = 3, a_1 = 5, a_2 = 8,$ $\quad a_n = a_{n-1} + a_{n-2} - a_{n-3}$ $(n \geq 3)$
10. $a_0 = 0, a_1 = -3, a_2 = -2,$ $\quad a_n = a_{n-1} + a_{n-2} - a_{n-3}$ $(n \geq 3)$
11. $a_0 = 3, a_1 = 2, a_2 = 5,$ $\quad a_n = a_{n-1} + a_{n-2} - a_{n-3}$ $(n \geq 3)$
12. $a_0 = 1, a_1 = 1, a_2 = 0, a_3 = 2,$ $\quad a_n = a_{n-3}$ $(n \geq 4)$

(Answers appear at the bottom of the back page.)

[2] In each of the following, find the generating function $\Phi(x) = \sum_{n\geq 0} a_n x^n$ where a_n is the number of k-element subsets σ of $\{1, 2, \ldots, n\}$ satisfying the given restrictions. (As a thought exercise, consider a_0 carefully in each case: the empty set has only one subset and this may satisfy the given conditions or it may not. Use the negation of the conditions, if necessary, to help decide.)

(a) $k = 3$ and no two elements of σ differ by less than three

(b) k arbitrary (i.e., sum over all k) and all elements of σ are even

(c) $k = 10$ and the i^{th} smallest element of σ is congruent to i modulo 3

(d) k arbitrary and no two elements of σ sum to n

[3] In these problems, we consider colored tilings of an n-board, a horizontal sequence of n unit squares. We will tile our board with colored squares and colored dominoes (these covering exactly two consecutive positions on our n-board). The parameters for each of the problems below are the following

symbol	name	meaning
k_1	square colors	number of available colors for square tiles
k_2	domino colors	number of available colors for domino tiles
Δ_{ss}	square-square flag	Δ_{ss} is TRUE if consecutive squares must have distinct colors
Δ_{sd}	square-domino flag	Δ_{sd} is TRUE if any domino appearing next to a square must have a color distinct from the color of that square
Δ_{dd}	domino-domino flag	Δ_{dd} is TRUE if consecutive dominoes must have distinct colors

(In all cases, assume that the domino colors form a subset of the square colors. For example, $k_1 = 3$, $k_2 = 2$ can be correctly interpreted as saying that there are red, white and blue squares to choose from as tiles, but only red and white dominoes.)

For each of the following choices of parameters,

- find the initial conditions (how many?)
- find a linear recurrence relation
- find a rational expression for the generating function $\sum_{n\geq 1} a_n x^n$ where a_n denotes the number of solutions on an n-board.

(a) $k_1 = k_2 = 2$, $\Delta_{ss} = T$, $\Delta_{sd} = T$, $\Delta_{dd} = T$

(b) $k_1 = 3$, $k_2 = 3$, $\Delta_{ss} = T$, $\Delta_{sd} = T$, $\Delta_{dd} = T$

(c) $k_1 = 3$, $k_2 = 2$, $\Delta_{ss} = T$, $\Delta_{sd} = T$, $\Delta_{dd} = T$

(d) $k_1 = 2$, $k_2 = 2$, $\Delta_{ss} = T$, $\Delta_{sd} = T$, $\Delta_{dd} = F$

(e) $k_1 = 2$, $k_2 = 2$, $\Delta_{ss} = T$, $\Delta_{sd} = F$, $\Delta_{dd} = F$

(f) $k_1 = 3$, $k_2 = 3$, $\Delta_{ss} = T$, $\Delta_{sd} = T$, $\Delta_{dd} = F$

(g) $k_1 = 3$, $k_2 = 3$, $\Delta_{ss} = F$, $\Delta_{sd} = T$, $\Delta_{dd} = T$

(h) $k_1 = 4$, $k_2 = 2$, $\Delta_{ss} = F$, $\Delta_{sd} = T$, $\Delta_{dd} = F$

Answers to Question [1]: a-8, b-4, c-6, d-2, e-12, f-5, g-9, h-11, i-3, j-10, k-1, l-7.