Combinatorics
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A Crash Course in Basic Math Notation

Mathematicians use a very compact, standard notation for the statements they make. These symbols are standardized over many years and it is not reasonable to make up your own notation or to invent your own interpretations of the symbols below. Each has a very precise meaning.

Set Theory and Logic

Logic		Set Theory	
AND	$p \wedge q$	set intersection	$A \cap B$
OR	$p \vee q$	set union	$A \cup B$
NOT	$\neg p$	set complement	$\bar{A}$
IMPLIES	$\Rightarrow$	subset relation	$\subseteq$
LOGICAL EQUIVALENCE	$\Leftrightarrow$, $\equiv$	set equality	$A = B$

In mathematics, we almost exclusively deal with *logical statements*, statements p that are either TRUE or FALSE, and *propositional functions*: statements $p(x)$ or $p(x, y, z, \ldots)$ about certain variables $x, y, z, \ldots$ which evaluate to logical statements for every possible acceptable value of the variables x, y, z, ... For example, the statement $-2 > 0$ is a logical statement (it is FALSE, I think) and the statement $(x > y) \Rightarrow (x^2 > y^2)$ is a propositional function which is TRUE for $x = 4, y = 2$ and for $x = -5, y = -2$ but FALSE for $x = -3, y = -4$.

The fundamental building blocks of mathematics are sets (collections of objects) and set membership. We write $a \in S$ if an element a belongs to a set S and we write $a \notin S$ as an abbreviation for $\neg(a \in S)$.

We can also take products of sets. The Cartesian product, written $A \times B$, of two sets A and B consists of all ordered pairs (a, b) where a is chosen from A and b is chosen from B. We can also form the set of all ordered triples (a, b, c) where $c \in C$ by taking the product $A \times B \times C$ and so on. In general, if $A_1, A_2, \ldots, A_n$ are all sets, then their Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n$$

consists of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for all i. For example,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

consists of all n-tuples of real numbers and inherits a geometric and vector space structure widely known as “n-dimensional real space”.

Quantifiers

Mathematical statements or theorems are often of the form "every object x of some certain type in set S satisfies some property $p(x)$" or "some object x of some certain type in set T satisfies some property $q(x)$." For example, the following are logical statements:

$$\forall x \in \mathbb{R} \ \ [(x > 2x) \Rightarrow (x > 3x)] ,$$

$$\exists x \in \mathbb{R} \ \ \left[(x^2 = x) \vee (x + 1 = x)\right]$$

and

$$\forall x \in \mathbb{Z} \ \ \left[(x + 0 = x) \wedge (x^2 > 0)\right] .$$

(The first two are TRUE while the third is FALSE.) Here, we have introduced the universal quantifier $\forall$ (read "for all" or "for every") and the existential quantifier $\exists$ ("there exists" or "there is").

The quantifier $\forall$ turns a propositional function into a logical statement. The newly formed logical statement $\forall x \in S \ (p(x))$ is TRUE only if the propositional function $p(x)$ is TRUE for every possible assignment of x from the set S. Even if $p(x)$ fails just once for some x in S, the statement $\forall x \in S \ (p(x))$ evaluates to FALSE.

Likewise, the quantifier $\exists$ turns a propositional function into a logical statement. The logical statement $\exists x \in S \ (p(x))$ is TRUE as long as the propositional function $p(x)$ is TRUE for at least one assignment of x from the set S. The only way the statement $\exists x \in S \ (p(x))$ becomes FALSE is if the propositional function $p(x)$ evaluates to a FALSE logical statement for every possible choice of x in S.

Now it is an easy thought exercise to verify the following negation rules:

$$\neg \left[\forall x \in S \ (p(x))\right] \ \equiv \ \exists x \in S \ (\neg p(x)),$$

$$\neg \left[\exists x \in S \ (p(x))\right] \ \equiv \ \forall x \in S \ (\neg p(x)).$$

I've also used a less common symbol, !, for uniqueness. The string of symbols $\exists ! x \in S \ldots$ is to be read "there exists a unique x in set $S \ldots$". This logical statement is TRUE provided an x exists in S for which the subsequent propositional function(s) hold(s) TRUE but **only one** such x exists. For example, a relation $f \subseteq A \times B$ is a *function* from A to B provided each member of a occurs exactly once as the first coordinate of some ordered pair in the set f:

$$\forall a \in A \ \exists ! b \in B \ \ ((a, b) \in f) .$$

Truth Tables

Let me finish by giving precise definitions of the logical and set operations defined at the beginning of this handout.

AND		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

OR		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

NOT	
p	$\neg p$
T	F
F	T

IMPLIES		
p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T