Combinatorics, D Term, 2008
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Combinatorics Assignment 7

Due Date: Monday, April 28, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep in mind Dr. Martin's rules for Combinatorics assignments.

1. What is the average number of ones in a binary string of length n having no two consecutive ones?

   [I.e., let $\mathcal{S}_n$ denote the collection of $\{0,1\}$-strings of length n which do not contain the substring 11. For each $n \geq 1$, find the average number of 1's in a string chosen uniformly at random from $\mathcal{S}_n$.]

2. Use the techniques of Section 3.2 in the blue notes to solve Exercise 2 on p81.

3. Recall the Fibonacci numbers F_n defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Now let

   $$f_n = \begin{cases} F_n, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

   Find a rational function expression for the generating function $\sum_{n\geq 0} f_n x^n$.

   [HINT: Recall that the generating function for the Fibonacci numbers is

   $$F(x) = x/(1 - x - x^2).$$

   Now consider $\frac{1}{2}\left(F(x) + F(-x)\right)$. This method of *series bisection* is a standard tool in combinatorial enumeration.]

4. A binary tree is said to be *extended* if at each vertex, the two subtrees above it are either both empty or both non-empty. By elementary means, construct (i.e., draw on paper) all extended binary trees on seven vertices or less.

   Based on your experiment, make a conjecture about the number of extended binary trees on n vertices.

5. If $T(x)$ is the generating function for extended binary trees, as defined in the previous question, show that

   $$T(x) = x\left\{1 + (T(x))^2\right\}$$

   and hence prove (or disprove!) the conjecture that you made in the previous exercise.