Combinatorics, D Term, 2008
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April 8, 2008

Combinatorics Assignment 5

DUE DATE: Thursday, April 17, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep in mind Dr. Martin's rules for Combinatorics assignments.

Please complete the following five problems:

1. In this problem (as throughout this assignment), our weight function is the length of a binary string.

   (a) Determine the generating function $\Phi(x)$ for the set of all binary strings in which every block of 0's has even length. I.e., find polynomials $P(x)$ and $Q(x)$ for which $\Phi(x) = \frac{P(x)}{Q(x)}$.

   (b) Use part (a) and the Sum Lemma to determine the generating function for all binary strings that have at least one odd-length block of 0's.

2. For $n \geq 0$, let a_n be the number of $\{0,1\}$-strings of length n in which each block of zeros has even length and each block of ones has odd length. Show that the generating function for these strings is

$$\sum_{n\geq 0} a_n x^n = \frac{1+x-x^2}{1-2x^2-x^3+x^4}.$$

3. For $n \geq 0$, let b_n be the number of $\{0,1\}$-strings of length n which do not contain 010 as a substring. Determine the generating function $\Phi(x) = \sum_{n\geq 0} b_n x^n$.
   [HINT: One way to do this is to use a modification of the block decomposition and first partition this set into: those strings containing no zeros; those that begin with a zero; and, those beginning with a one and containing at least one zero.]

4. For $n \geq 0$, let c_n be the number of $\{0,1\}$-strings of length n in which each block of ones is preceded by a block of zeros at least as long as that block of ones. Determine the generating function $\Phi(x) = \sum_{n\geq 0} c_n x^n$.

   For example, the following strings are in our language $\mathcal{L}$: 01000011, 000000; but these strings are **not**: 00110001111, 1010001.

5. Let d_n be the number of $\{0,1\}$-strings of length n having no block of exactly two ones. (I.e., 01111001 and 11110111 are included, but 11, 00011 and 0101011000 are not.) By first finding a rational expression for the generating function $\sum_{n\geq 0} d_n x^n$, find a recurrence relation for the coefficients d_n and sufficient initial conditions to determine the entire sequence. (Can you explain the recurrence from first principles? If so, come talk to me.)